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H_∞ Filter Design for Vehicle Tracking Under Delayed and Noisy Measurements

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***Abstract*— In many intelligent vehicles applications tracking plays an important role. This paper considers tracking of a vehicle under delayed and noisy measurements. For this purpose we design an H_∞ optimal filter for linear systems with time delays in the state and output variables. By using the duality between filtering and control, the problem at hand is transformed to a robust controller design for systems with time delays. The skew Toeplitz method developed earlier for the robust control of infinite dimensional systems is used to solve the H_∞ filtering problem. The results are illustrated with simulations and effects of the time delay on the tracking performance are demonstrated.**

I. INTRODUCTION

This paper deals with an important aspect of the tracking problems appearing in intelligent vehicles applications, namely state estimation under delayed and noisy measurements. An example for the problem studied here is illustrated in Figure 1, where a target is moving according to a certain known dynamical equations (position, velocity and acceleration representing the state $x(t)$) with unknown input $w(t)$. Suppose that the position of the target is the measured variable, but the measurement is noisy and it reaches the processing unit with a certain time delay, which may be due to physical distance between the target and the processing unit and/or due to restrictions imposed by the communication channels. The processing unit receives the signal $y(t) = Cx(t-h)+v(t)$, (where C is a constant matrix, $h > 0$ is the delay amount, and $v(t)$ is the measurement noise) and generates an estimate $\hat{z}(t)$ of the current position $z(t) = Cx(t)$.

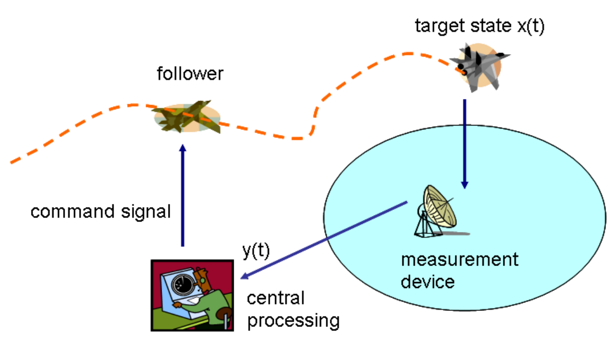

Fig. 1. Tracking Problem

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Estimate of the target vehicle's position, $\hat{z}(t)$, may then be used to give a command signal to the follower vehicle, which may be required to follow the path traveled by the target vehicle, or to reach the target within a certain desired time interval. In this paper we will not deal with what the follower does based on the command signal received from the central processing unit. Rather, we will concentrate on how well the position, $z(t)$, can be estimated and discuss the effect of time delay on best achievable tracking error.

In the literature many techniques have been developed to solve the above problem within the framework of H_∞ filtering. These methods primarily depend on the dynamical model of the target. Our goal is to solve the H_∞ optimal filtering problem without approximations of the time delay. Previous works mostly dealt with designing observers for time delayed state variables, i.e. time delays are in the state dynamics, [1], [2], [3], [7]. Linear functional state observers with delay and stability conditions are given in [1] for delay dependent cases. For systems with delay in the state and the output an H_∞ filter design, which is of the Luenberger observer type is presented in [5] depending on a newly designed version of the bounded real lemma for time delay systems. A robust H_∞ filtering method is proposed in [6] for linear continues systems with time varying delay. The filter is a linear observer type and guarantees that L_2 induced norm from exogenous signal to estimation error is less than a prescribed value. A number of Linear Matrix Inequalities (LMIs) are solved to obtain the filter. Another filtering method that uses LMI solutions for time varying multiple delays in state variables is given in [7] which solves robust L_2-L_∞ filtering problem guaranteeing a prescribed energy to peak noise attenuation level for uncertainties and time delays. A different method of H_∞ observer design is proposed in [2] which studies a linear system with multiple delays in state and output. Another method of designing an observer is given in [3]; again, it involves LMIs. We should indicate that most of the above mentioned techniques involving LMIs are suboptimal in the sense that the filter can be obtained under the condition that the LMIs are solvable. In most situations the optimal performance level cannot be achieved. Besides the frequency domain method proposed in this paper, there are some time domain state-space based techniques leading to optimal H_∞ filters, see e.g. [10], [11]. In [11] a lifting technique is used to solve the associated Nehari problem (see Section II below). In [10], Mirkin solves the problem by parameterizing all solutions of the non-delayed problem and finding the ones which solve the delayed problem. This approach involves solving Riccati equations and checking a

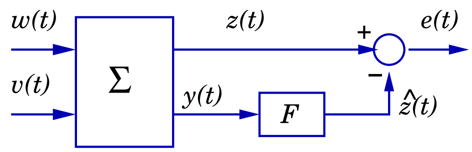

Fig. 2. Dynamic System Model for Estimation

spectral radius condition. Among all available methods for the solution of the H_∞ *suboptimal* filtering problem under delayed and noisy measurements, Mirkin's approach [10] is the simplest. Moreover, his "central" filter's performance can get arbitrarily close to the optimum.

In this paper, using the frequency domain representations, we provide an alternative method to compute the H_∞ *optimum* filter directly. First, by using the duality between filtering and control, the problem at hand is transformed to a robust controller design for systems with time delays. The skew Toeplitz method developed earlier for the robust control of infinite dimensional systems, [4], [12], [8], is used to solve the H_∞ *optimal* filtering problem.

Next section describes the problem and propose a new filter design technique using the duality between filtering and control. Section III gives an illustrative example to demonstrate the solution method as well as the effect of time delay on the tracking performance. Concluding remarks are made in the last section.

II. PROBLEM FORMULATION AND METHODOLOGY

Consider the dynamical system (Σ) shown in Figure 2 with time delays in state and output. The objective of this paper is to design a filter F so that the error e is small in the H_∞ sense, i.e. the L_2 induced gain from $[w \quad v]^{\mathrm{T}}$ to e is small.

A. Problem Definition

Consider the linear time-delay system which is shown as Σ in Figure 2:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - h_1) + Bw(t) \tag{1}$$

$$y(t) = C_0 x(t) + C_1 x(t - h_2) + Dv(t) \tag{2}$$

$$z(t) = Lx(t) \tag{3}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^p$ is output vector, $w(t) \in \mathbb{R}^q$ and $v(t) \in \mathbb{R}^q$ are process noise and measurement noise vectors respectively. Time delays h_1 and h_2 are assumed to be known. The matrices A_0, A_1, B, C_0, C_1, D and L are also known. In this case the transfer matrices from disturbances to state and output are found from the relations

$$X(s) = R(s)BW(s) \tag{4}$$

where $R(s) := (sI - A_0 - A_1 e^{-h_1 s})^{-1}$. Then,

$$Y(s) = (C_0 + C_1 e^{-h_2 s})R(s)BW(s) + DV(s) \tag{5}$$

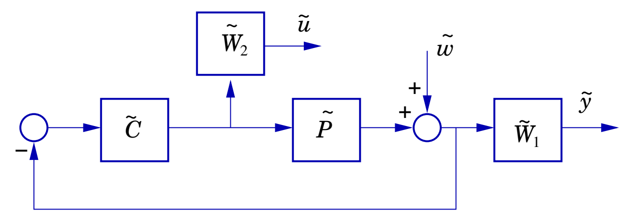

Fig. 3. Feedback Control System

We seek a filter such that the estimation error e is small in the H_∞ sense,

$$e(t) = z(t) - \hat{z}(t). \tag{6}$$

In the frequency domain, we have

$$E(s) = Z(s) - \hat{Z}(s) = LX(s) - F(s)Y(s) \;= \\ (L - F(s)(C_0 + C_1 e^{-h_2 s}))R(s)BW(s) - F(s)DV(s) \tag{7}$$

Assumption: In order to simplify the exposition we assume

$$C_1 = L = C \qquad C_0 = 0. \tag{8}$$

Otherwise the inner-outer factorization techniques mentioned in [8] can be used here.

With the above assumption the estimation error takes the form

$$E(s) = U(s)(1 - F(s)e^{-h_2 s})W(s) - F(s)DV(s) \tag{9}$$

where

$$U(s) = C(sI - A_0 - A_1 e^{-h_1 s})^{-1}B. \tag{10}$$

Let us now assume that the measurement noise v is generated by a known coloring filter W_v, i.e. $V(s) = W_v(s)\widehat{V}(s)$, where $\widehat{v}$ is an unknown finite energy signal. Similarly, let w be an unknown finite energy signal. Then the L_2 induced norm from external signals w and $\widehat{v}$ to the error e is

$$\begin{aligned} \gamma &= \|(1 - F(s)e^{-h_2 s})U(s) \qquad - F(s)DW_v(s)\|_\infty \\ &= \sup_{\widehat{v}, w \neq 0} \frac{\|e\|_2}{\left\| \begin{bmatrix} w \\ \widehat{v} \end{bmatrix} \right\|_2} \end{aligned} \tag{11}$$

Clearly the following two conditions must be satisfied in order to have a finite γ:

$$\begin{aligned} &F(s) \text{ is stable, and} \\ &(1 - F(s)e^{-h_2 s})U(s) \text{ is stable} \end{aligned} \tag{12}$$

B. H_∞ Control Problem

The standard H_∞ control problem associated with a stable plant $\widetilde{P}$ shown in Figure 3 can be defined as follows.

Transfer functions from the disturbance $\widetilde{w}$ to $\widetilde{y}$ and $\widetilde{u}$ are:

$$T_{\widetilde{w}\to\widetilde{y}} = \widetilde{W}_1(1 + \widetilde{P}\widetilde{C})^{-1}$$

$$T_{\widetilde{w}\to\widetilde{u}} = -\widetilde{W}_2\widetilde{C}(1 + \widetilde{P}\widetilde{C})^{-1} \tag{13}$$

The optimal H_∞ controller design problem is:

$$\begin{array}{ll} \text{minimize} & \gamma \\ \text{subject to} & (\widetilde{P},\widetilde{C}) \text{ is stable, and} \end{array} \qquad (14)$$

$$||T_{\widetilde{w}\rightarrow\left[\begin{array}{c}\widetilde{y}\\ \widetilde{u}\end{array}\right]}||_\infty \leq \gamma \qquad (15)$$

In order to find the smallest (i.e. optimal) γ, following is solved:

$$\inf_{\widetilde{Q}\in H_\infty} \|\widetilde{W}_1(1-\widetilde{P}\widetilde{Q}) \qquad -\widetilde{W}_2\widetilde{Q}\|_\infty \qquad (16)$$

The free parameter $\widetilde{Q}$ is obtained from the controller

$$\widetilde{C} = \frac{\widetilde{Q}}{1-\widetilde{P}\widetilde{Q}} \qquad \widetilde{Q} = \frac{\widetilde{C}}{1+\widetilde{P}\widetilde{C}}.$$

The important point throughout this work is that (16) is same problem with (11) provided that the following dualities are established:

$$\begin{aligned} \widetilde{W}_1(s) &= U(s) = C(sI - A_0 - A_1 e^{-h_1 s})^{-1}B \\ \widetilde{W}_2 &= I \\ \widetilde{P}(s) &= e^{-h_2 s}D^{-1}W_v^{-1}(s) \\ \widetilde{Q}(s) &= F(s)DW_v(s) \end{aligned} \qquad (17)$$

Thus, the result of the H_∞ optimal control problem, $\widetilde{Q}$, gives the H_∞ optimal filter F.

C. Solution of the H_∞ Control Problem

It is clear from (17) that the H_∞ control problem defined above involves infinite dimensional weight $\widetilde{W}_1 = U(s)$ and a stable plant with time delay. We now present the solution to the above control problem for the case $A_1 = 0$ (or $h_1 = 0$) and $h_2 \neq 0$. It is possible to solve the problem when $h_1 \neq 0$ and $h_2 = 0$; but an exact optimal solution is difficult to obtain when both delays are non-zero, in such a case one may have to try finding approximate solutions.

The optimal H_∞ controller satisfying (16) is designed in [4] and it is given in the form of:

$$\widetilde{C}_{opt}(s) = E_{\gamma_0}(s)\frac{N_0(s)^{-1}F_{\gamma_0}(s)L(s)}{1+m_n(s)F_{\gamma_0}(s)L(s)} \qquad (18)$$

where $m_n(s) = e^{-h_2 s}$, $N_o(s) = D^{-1}W_v^{-1}(s)$, $E_{\gamma_0}(s) = U_{\gamma_0}(s)U_{\gamma_0}(-s) - 1$, with $U_{\gamma_0}(s) = U(s)/\gamma_0$, and $F_{\gamma_0}(s)$ and $L(s)$ are rational functions determined from the problem data, see [4], [12]. Then, the desired filter is obtained as

$$F(s) = D^{-1}W_v^{-1}\widetilde{Q}_{opt} = D^{-1}W_v^{-1}\widetilde{C}_{opt}(1+\widetilde{P}\widetilde{C}_{opt})^{-1}$$

For $\widetilde{W}_1 = U(s) = C(sI - A)^{-1}B$ (i.e. $h_1 = 0$), we have the following structure for the optimal filter:

$$F(s) = \frac{(U_{\gamma_0}(s)U_{\gamma_0}(-s) - 1)\ F_{\gamma_0}(s)L(s)}{1+e^{-h_2 s}F_{\gamma_0}(s)L(s)U_{\gamma_0}(s)U_{\gamma_0}(-s)}. \qquad (19)$$

In the next section we illustrate the computation this filter with an example.

III. NUMERICAL EXAMPLE

Consider the system (1) with the assumptions (8) and $A_1 = 0$ (i.e. $U(s)$ is rational). Then, we have

$$\dot{x}(t) = Ax(t) + Bw(t) \qquad (20)$$

$$y(t) = Cx(t-h_2) + Dv(t) \qquad (21)$$

$$z(t) = Cx(t) \qquad (22)$$

$x(t)$ is the state vector of the target vehicle and it is composed of

$$x(t) := \begin{pmatrix} x_p \\ x_v \\ x_a \end{pmatrix} \qquad \begin{array}{l} x_p : position \\ x_v : velocity \\ x_a : acceleration \end{array} \qquad (23)$$

The corresponding matrices are

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\varepsilon \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \qquad D = 1 \qquad (24)$$

Here ε is a parameter which determines how much the initial value of the acceleration impacts the system dynamics. We arbitrarily take it as $\varepsilon = 2$. Now, with the above we have

$$U(s) = \frac{1}{s^2(s+\varepsilon)}.$$

The above system describes a moving vehicle whose acceleration depends on $w(t)$, considered as an unknown finite energy signal. In all simulations below $w(t)$ is as shown in Figure 4.

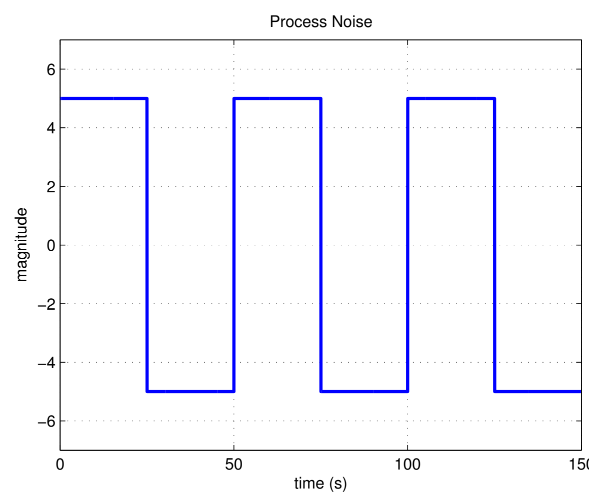

Fig. 4. Process Noise, $w(t)$

The model above is extended to two-dimensional space by repeating it and the corresponding filter for the x-and-y directions independently. So the trajectory to be tracked is shown in two-dimensional space in Figure 5. The disturbance in the acceleration ($w(t)$ shown above is repeated in x and y directions) leads to maneuvers as seen in the figure.

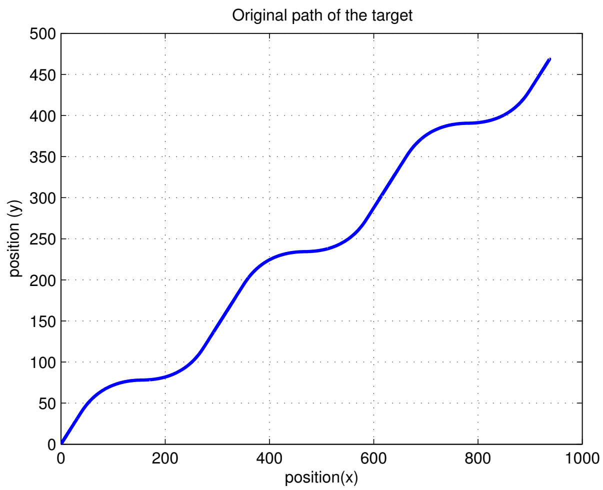

Fig. 5. Original trajectory

Case 1. Let $DW_v(s) = 1$, and $h_2 = 1$. The signal shown in Figure 6 is taken as the measurement noise in both x and y directions.

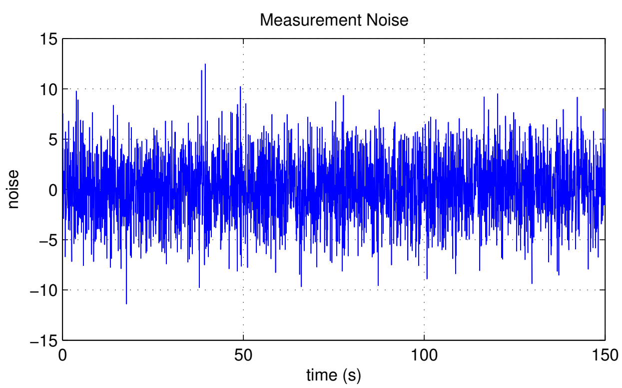

Fig. 6. Measurement Noise, Case 1.

For these numerical values, the functions $F_\gamma(s)$ and $L(s)$ necessary to obtain the filter from (19) are found by the help of MATLAB:

$$L(s) = -\frac{s^2 + 2.35s + 0.72}{s^2 - 2.35s + 0.72}$$

$$F_\gamma(s) = \frac{s^2(s^2 - \varepsilon^2)}{0.655s^4 + 3.183s^3 + 5.104s^2 + 3.22s + 0.99}$$

with $\gamma = 1.526$. The final form of the filter (19) is

$$F(s) = \frac{\gamma R_1(s)}{1 + R_1(s)R_2(s)} \quad (25)$$

where $R_1(s)$ and $R_2(s)$ are Infinite Impulse Response (IIR) and Finite Impulse Response (FIR) filters respectively, i.e. impulse response of R_2 is zero outside the time interval $[0\ , h_2]$. For the above numerical values of the problem we have

$$R_1(s) \approx \frac{s^2 + 2.35s + 0.72}{s^2 + 2.5s + 1.1} \quad (26)$$

$$\begin{aligned} R_2(s) \quad &\approx \frac{0.655e^{-s}}{s^6 - 4s^4 + 0.43} \quad (27) \\ &+ \frac{0.006s^5 - 0.03s^4 + 0.11s^3 - 0.33s^2 + 0.66s - 0.66}{s^6 - 4s^4 + 0.43} \end{aligned}$$

Time domain simulations have been performed for this system with different disturbance signals. Following figures show the estimation performance of filter against time delay.

Figure 7 shows the error in the output, namely the difference $z(t) - y(t) = C(x(t) - x(t - h_2)) - v(t)$. Effect of time delay is obvious in the figure. If the time delay h_2 was zero, then this signal would be equal to the measurement noise $-v(t)$, see Figure 6. Therefore, the deviation of $z(t) - y(t)$ from $-v(t)$ shows how difficult the filtering problem is (the problem is not just a simple noise elimination problem).

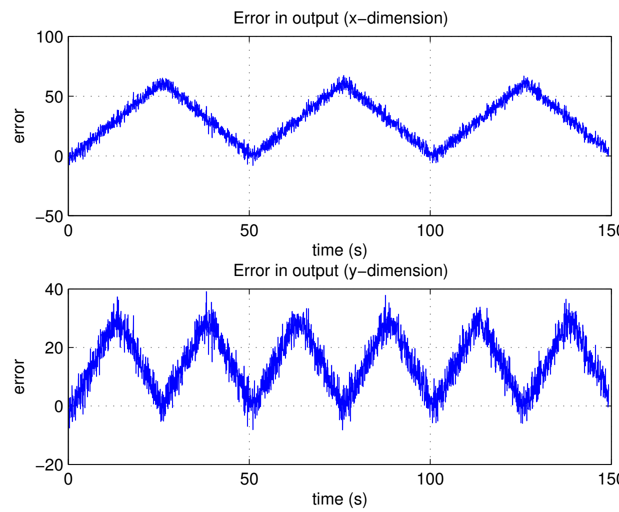

Fig. 7. $z(t) - y(t)$ for Case 1.

The performance of filter is shown in Figure 8. It illustrates the estimation error $z(t) - \hat{z}(t)$ along the path. Error caused by time delay is corrected by filter and just a noise like characteristics similar to measurement noise is left as the error.

We have also applied the method of Mirkin, [10] on the same problem. Note that in [10] we have to choose a γ which is greater than the optimal value $\gamma_o = 1.526$. Then a central suboptimal filter is designed. In order to compare the performance of the optimal filter and the "near optimal" filter of [10] we show both estimation errors in Figure 8 (where dark lines correspond to the result of the filter of [10]). It looks like the filter of [10] can eliminate the measurement noise better, but on the average it leads to a larger error.

We have also implemented a standard Kalman filter for the discretized delayed system model (state space has expanded by sampling 20 times during a one delay time period). The resulting error is shown in Figure 9. We see that Kalman filter can eliminate the noise, but it cannot reduce the effect of time delay.

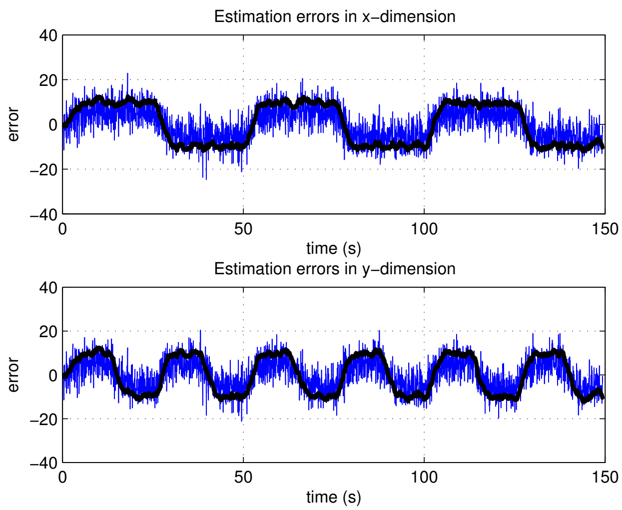

Fig. 8. Estimation Errors, Case 1.

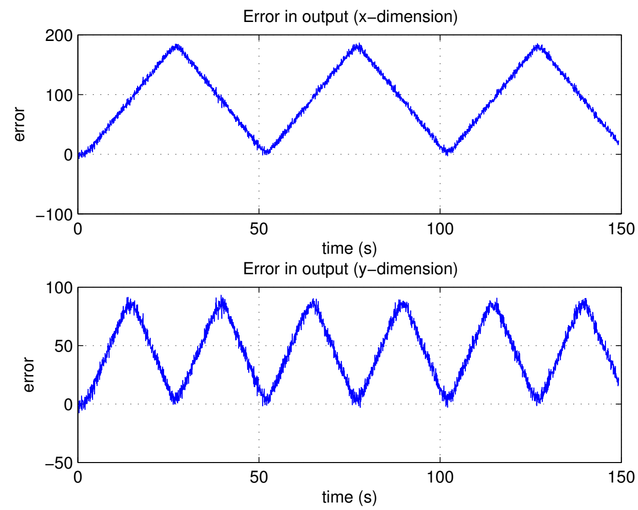

Fig. 10. $z(t) - y(t)$, Case 2.

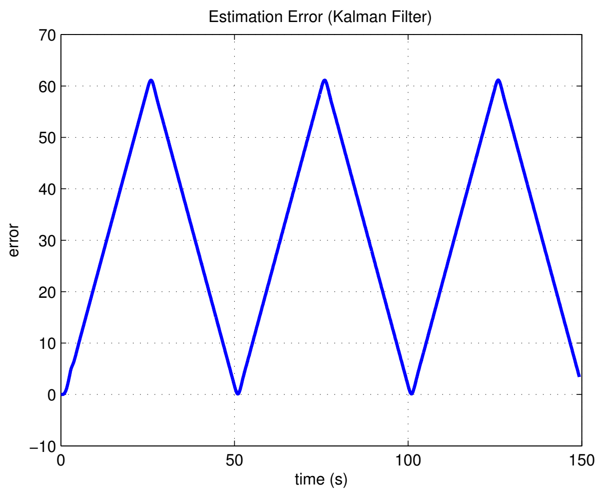

Fig. 9. Estimation Error with Kalman Filter, Case 1.

Case 2. Let $DW_v(s) = 1$, and $h_2 = 3$. The resulting filter expression is the same as (25), where this time we have $\gamma = 3.654$ and

$$R_1 \approx \frac{s^2 + 2.264s + 0.529}{s^2 + 1.334s + 1.642} \tag{28}$$

$$\begin{aligned} R_2 &\approx \frac{0.274e^{-3s}}{s^6 - 4s^4 + 0.075} \\ &+ \frac{1.37s^5 - 3.12s^4 + 1.23s^3 - 1.23s^2 + 0.81s - 0.23}{s^6 - 4s^4 + 0.075} \end{aligned} \tag{29}$$

The above time domain simulations are repeated for this case. Figure 10 and Figure 11 are the errors before the filter, $z(t) - y(t)$ and after filter, $z(t) - \widehat{z}(t)$ respectively.

As before we also provide the result obtained using [10], in Figure 11 as dark line. We see that in this case, the average value of the error obtained using the filter proposed in [10] is about the same as the average value of the error obtained using the optimal filter derived here.

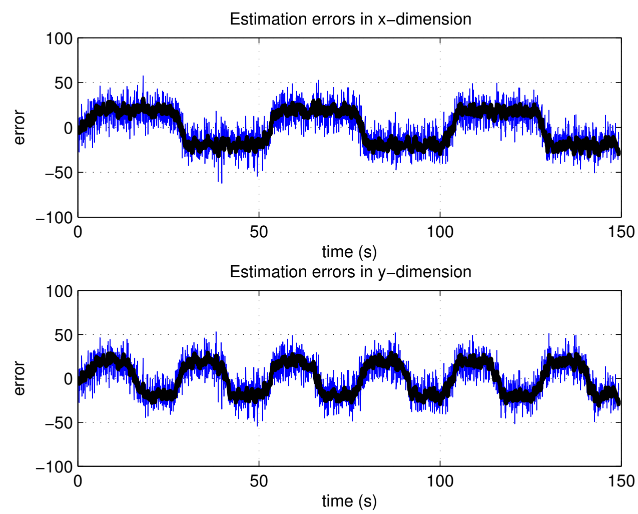

Fig. 11. Estimation Errors, Case 2.

Above examples have shown that the errors due to time delay are eliminated successfully by using the H_∞ optimal filter derived here. On the other hand, the effect of measurement error seems to be there. In order to reduce the effect of the measurement error we may consider using a weight $W_v(s)$ which generates $v(t)$. This is the next study case.

Case 3. $h_2 = 3$ and $DW_v(s) = \frac{10s+1}{s+10}$. For this case we compute $\gamma = 4.188$. And the filter can again be put in the form of (25). Figure 12 shows the error in delayed state $z(t) - y(t)$, and Figure 13 is the estimation error of the filter $z(t) - \widehat{z}(t)$, using the method proposed here (blue line) and the method of [10] (dark line). By comparing these two graphs we observe that the filter eliminates the effect of time delay and it reduces the noise by about a factor of two.

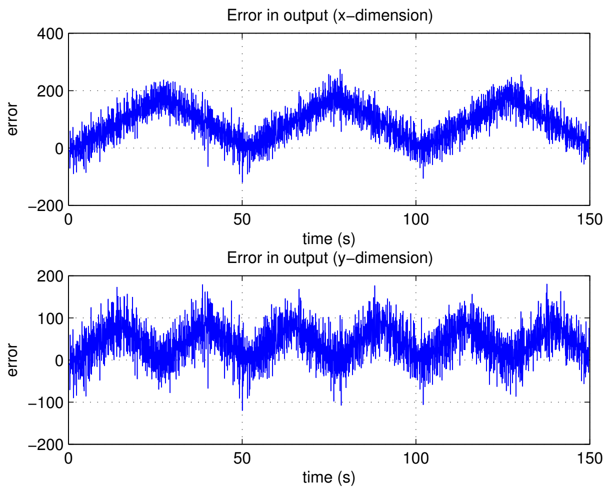

Fig. 12. $z(t) - y(t)$, Case 3.

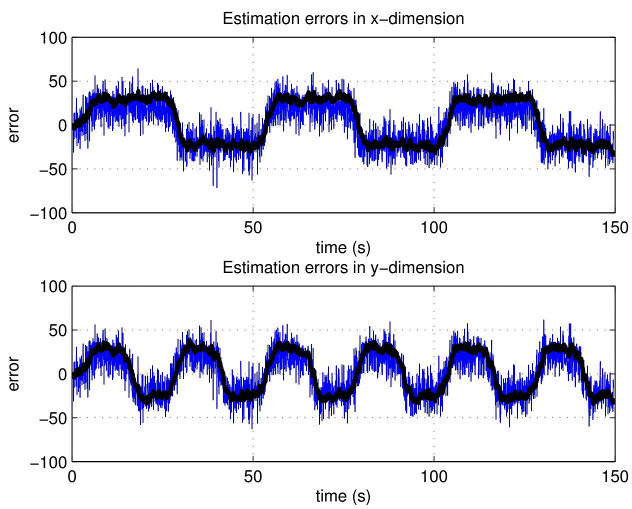

Fig. 13. Estimation Errors, Case 3.

IV. CONCLUDING REMARKS

In this paper, by using the duality between filtering and control, we have illustrated that the earlier methods developed for the robust control of infinite dimensional systems solve the H_∞ filtering problem appearing in target tracking problems under delayed and noisy measurements.

The structure of the filter designed is very simple (25); one needs to compute the performance level γ, and two functions $R_1(s)$ and $R_2(s)$. In our numerical examples $R_1(s)$ was a low order rational function and $R_2(s)$ was an FIR filter whose coefficients can be computed explicitly using the formulae given in Section II, and the results of [4], [12].

Simulations show that compared to the method proposed in [10], the H_∞ optimal filter (25) results in more noisy estimation errors. This is due to the gain of the optimal filter at $s = +\infty$, i.e., in Case 1 and 2 we have $F(\infty) = \gamma$, which means that the high frequency component of the noise is amplified/attenuated by a factor of γ. Whereas the central suboptimal filter of [10] is always strictly proper, hence high frequency noise is always filtered. Similarly for the Kalman filter: high frequency noise is filtered, but the effect of the time delay is there.

For the case where $h_1 \neq 0$ and $A_1 \neq 0$ we may have to approximate the function $U(s)$ by a rational function so that this approach works. The results for this situation will be reported elsewhere due to page restrictions.

REFERENCES

[1] M. Darouach, "Linear Functional Observers for Systems with Delays in State Variables" *IEEE Trans. on Automatic Control*, Vol. 46, 2001, pp. 491–496.

[2] A. Fattouh, O. Sename and J. M. Dion, "Robust Observer Design for Linear Time-Delay Systems: A Riccati Equation Approach" Kybernetika (Prague), 1999.

[3] A. Fattouh, O. Sename and J. M. Dion, "A LMI Approach to Robust Observer Design for Linear Time-Delay Systems" *Proc.* 39^{th} *IEEE CDC*, Sydney, Australia, December 2000, pp. 1495–1500.

[4] C. Foias, H. Özbay, A. Tannenbaum, *Robust Control of Infinite Dimensional Systems: Frequency Domain Methods*, LNCIS No. 209, Springer-Verlag, London, 1996.

[5] E. Fridman and U. Shaked, "A New H_∞ Filter Design for Linear Time Delay Systems " *IEEE Trans. on Signal Processing*, Vol. 49, 2001, pp. 2839–2843.

[6] E. Fridman, U. Shaked and L. Xie, "Robust H_∞ Filtering of Linear Systems with Time Varying Delay" *IEEE Trans. on Automatic Control*, Vol. 48, 2003, pp. 159–165.

[7] H. Gao and C. Wang, "Robust L_2-L_∞ Filtering for Uncertain Systems with Multiple Time-Varying State Delays" *IEEE Trans. on Circuits and Systems-I: Fundamental Theory and Applications*, Vol. 50, 2003, pp. 594–599.

[8] S. Gümüşsoy and H. Özbay, "Remarks on H_∞ Controller Design for SISO Plants with Time Delays" *Proceeding of the* 5^{th} *IFAC Symposium on Robust Control Design*, Toulouse, France, July 2006.

[9] T. D. Larsen, N. A. Andersen, O. Ravn and N. K. Poulsen "Incorporation of Time Delayed Measurements in a Discrete-time Kalman Filter" *Proc.* 37^{th} *IEEE Conference on Decision & Control* Tampa, Florida USA, December 1998, pp. 3972–3977.

[10] L. Mirkin, "On the extraction of dead-time controllers and estimators from delay-free parametrizations" *IEEE Trans. on Automatic Control* Vol. 48, 2003, pp. 543–553.

[11] K. M. Nagpal and R. Ravi, "H_∞ Control and Estimation Problems with Delayed Measurements: State Space Solutions" *SIAM J. Control Optim.* Vol. 35, 1997, pp. 1217–1243.

[12] O. Toker and H. Özbay, "H^∞ Optimal and suboptimal controllers for infinite dimensional SISO plants," *IEEE Transactions on Automatic Control*, vol. 40, 1995, pp. 751–755.