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Vehicle Positioning by a Confidence Interval Observer - Application to an Autonomous Underwater Vehicle

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Abstract— **State estimation methods allow the vehicle position and velocity to be reconstructed by combining information from sensors and vehicle model. From a security point of view, position and velocity have to be known with a high level of confidence in order, for example, to avoid vehicle collision. In this paper, a confidence interval observer is developed to enclose positioning variables with some confidence degree (or integrity level). For this purpose, the algorithm is divided in two parts. First, a predictor, based on the vehicle dynamics, is derived to estimate bounds on state variables with lower bounded integrity. Then, at each measurement time, confidence intervals from the sensors are combined with union and intersection operations to satisfy the integrity level. The shortest non-empty intervals are chosen among the safe intervals. Finally, to quantify the reliability of estimation, a security measure is defined by the probability of having one faulty estimation in some period of time and is related to the integrity level objective. This method is illustrated with simulation tests based on an autonomous underwater vehicle described by a nonlinear model.**

I. INTRODUCTION

State estimation is commonly used in vehicle positioning applications where velocity and position are reconstructed based on a vehicle model and on-line measurements. Accuracy is of course desirable, but in some applications, safety can be even more important. For instance, aircraft or train positioning requires a very high level of security so as to avoid collisions which could cause human and material loss.

In these applications, positioning information is usually given in the form of intervals bounding variables and parameters with some degree of confidence. In this context, interval algebra [6] has been used to build guaranteed intervals. However, guaranteed intervals can be a too idealistic assumption in some applications with critical safety requirements. For instance, the reliability of safety-related process can be assessed by a statistical measure named **SIL** (**S**afety **I**ntegrity **L**evel, [5]) and some vehicle applications (railway, aeronautics) are related to the highest level **SIL4**, i.e. 10^{-9} tolerated error per hour. In this context, recent work has been devoted to the manipulation of confidence intervals in a static configuration [2], [8].

In this study, a confidence interval observer is designed for vehicles described by a nonlinear dynamic model. Two parts are considered. First, between two sampling times, a predictor is built propagating state variable bounds. Model transformations are used in order to reduce the rate of increase of the interval size in the interval between two measurement times, where the only source of information is the uncertain model. Secondly, sensors are combined to give required confidence intervals. The integrity level objective is defined according to **SIL** by computing the probability of having one faulty estimation in some period of time. The chosen application is an autonomous underwater vehicle (AUV, [7]).

The paper is organized as follows. Section II presents the notations, the concept of a confidence interval and the basic operations - intersection and union - that can be carried out on the intervals. Section III describes the proposed confidence interval observer which is divided into a prediction step between two measurement times and a correction step at each sampling time. Section IV is devoted to the algorithm illustration. Finally, conclusions are drawn in Section V.

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II. PRELIMINARIES

We consider continuous-time nonlinear models associated to discrete-time measurements:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t),\mathbf{u}(t),\boldsymbol{\theta}) \qquad \mathbf{x}(0)=\mathbf{x}_0 \tag{1}$$

$$\mathbf{y}_k = C(t_k)\mathbf{x}(t_k) \tag{2}$$

where $\mathbf{x}(t)\in\Re^{n_x}$ is the vector of state variables, $\mathbf{u}(t)\in\Re^{n_u}$ is the vector of inputs, $\mathbf{y}_k\in\Re^{n_y}$ is the vector of measurements at the discrete time t_k, $\boldsymbol{\theta}$ is a parameter vector. The subvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ represent the measured and unmeasured state variables, respectively, i.e. $\mathbf{x}_k=\left[\mathbf{x}_{1,k}^T,\mathbf{x}_{2,k}^T\right]^T$.

Intervals are assumed for the initial state $\mathbf{x}_0$, the parameters $\boldsymbol{\theta}$ and the inputs $\mathbf{u}(t)$. Their confidence (or integrity) level is defined in the following way:

$$\begin{aligned} p\left(\mathbf{x}_0\in\left[\mathbf{x}_0^-,\mathbf{x}_0^+\right]\right) &\geq \beta_0 \\ p\left(\boldsymbol{\theta}\in\left[\boldsymbol{\theta}^-,\boldsymbol{\theta}^+\right]\right) &\geq \beta_{\boldsymbol{\theta}} \\ p\left(\mathbf{u}(t)\in\left[\mathbf{u}_k^-,\mathbf{u}_k^+\right]\right) &\geq \beta_{\mathbf{u}_k} \quad t_k\leq t\leq t_{k+1} \end{aligned} \tag{3}$$

Let $\mathbf{y}_{meas,k}\in\Re^{n_y}$ denote the sensor measurements. In a similar way, intervals are built such that the integrity level is given by:

$$p\left(\mathbf{x}_{1,k}\in\left[\mathbf{y}_{meas,k}^-,\mathbf{y}_{meas,k}^+\right]\right)\geq\beta_{\mathbf{y}_k} \tag{4}$$

Considering these notations, the confidence interval observer aims at computing upper $\mathbf{x}^+(t)$ and lower $\mathbf{x}^-(t)$ bounds of the state vector $\mathbf{x}(t)$ so that the related integrity satisfies the objective :

$$p\left(\mathbf{x}(t)\in\left[\mathbf{x}^-(t),\mathbf{x}^+(t)\right]\right)\geq\beta_{obj} \tag{5}$$

Moreover, a Boolean random variable $B_\bullet$ is related to each computed interval (parameters and variables), following the definition [1]:

$$\begin{aligned} B_a &= \text{true} \quad \text{if } a \in [a^-, a^+] \\ B_a &= \text{false} \quad \text{otherwise} \end{aligned} \tag{6}$$

From this definition, one can deduce :

$$p(B_a = \text{true}) = p(a \in [a^-, a^+]) = \beta_a \tag{7}$$

Intuitively, this means that, for infinitely repeated random events computing an interval on a, bounding will occur in the mean at the rate β_a.

Basic operations on these random variables can now be defined [1]. Let B_{a_1} and B_{a_2} denote two Boolean random variables deduced from two sensors measuring a with the intervals $[a_1^-, a_1^+]$ and $[a_2^-, a_2^+]$, and let β_{a_1} and β_{a_2} denote the lower bounds on the integrity levels, respectively.

$$\begin{aligned} p(B_{a_1} = \text{true}) &= p(a \in [a_1^-, a_1^+]) \geq \beta_{a_1} \\ p(B_{a_2} = \text{true}) &= p(a \in [a_2^-, a_2^+]) \geq \beta_{a_2} \\ p(B_{a_1} = \text{false}) &= p(a \notin [a_1^-, a_1^+]) \leq 1 - \beta_{a_1} \\ p(B_{a_2} = \text{false}) &= p(a \notin [a_2^-, a_2^+]) \leq 1 - \beta_{a_2} \end{aligned}$$

Intersection :

- If we consider independent random variables

$$\begin{aligned} & p\left(a \in [a_1^-, a_1^+] \cap [a_2^-, a_2^+]\right) = \\ & p\left(B_{a_1} = \text{ true} \quad \textit{and} \quad B_{a_2} = \text{ true}\right) \geq \beta_{a_1}\beta_{a_2} \end{aligned} \tag{8}$$

- If we consider dependent random variables

$$\begin{aligned} & p\left(B_{a_1} = \text{ true} \quad \textit{and} \quad B_{a_2} = \text{ true}\right) \geq \\ & p\left(B_{a_1} = \text{ true}\right) - p\left(B_{a_2} = \text{ false}\right) \geq \\ & \max\left(0, \beta_{a_1} + \beta_{a_2} - 1\right) \end{aligned} \tag{9}$$

Union :

- If we consider independent random variables

$$\begin{aligned} & p\left(a \in [a_1^-, a_1^+] \cup [a_2^-, a_2^+]\right) = \\ & p\left(B_{a_1} = \text{ true} \quad \textit{or} \quad B_{a_2} = \text{ true}\right) = \\ & 1 - p\left(B_{a_1} = \text{ false} \quad \textit{and} \quad B_{a_2} = \text{ false}\right) \geq \\ & 1 - (1 - \beta_{a_1})(1 - \beta_{a_2}) = \beta_{a_1} + \beta_{a_2} - \beta_{a_1}\beta_{a_2} \end{aligned} \tag{10}$$

- If we consider dependent random variables

$$p\left(B_{a_1} = \text{ true} \quad \textit{or} \quad B_{a_2} = \text{ true}\right) \geq \max\left(\beta_{a_1}, \beta_{a_2}\right) \tag{11}$$

These notions are now used in the confidence interval observer design.

III. Confidence Interval Observer

Based on the general model (1)-(2) and intervals for the initial state, the parameters, the inputs and the measurements (3)-(4), the confidence interval observer proceeds in two separate steps with respect to the measurement times:

A. Prediction step

Between two measurement times, the predictor propagates the intervals bounds on the state variables. If we assume bounded intervals at time t_k ($\mathbf{x}(t_k) \in [\mathbf{x}_P^-(t_k), \mathbf{x}_P^+(t_k)]$), then the predictor (x_P^-, x_P^+) has to be designed in such a way that it guarantees $\mathbf{x}(t) \in [\mathbf{x}_P^-(t), \mathbf{x}_P^+(t)]$ for $t \geq t_k$. Equivalently, the error vector $e(t) \in \Re^{2n_x}$ has to be positive, i.e.

$$e(t) = \begin{bmatrix} e^+(t) \\ e^-(t) \end{bmatrix} = \begin{bmatrix} \mathbf{x}_P^+(t) - \mathbf{x}(t) \\ \mathbf{x}(t) - \mathbf{x}_P^-(t) \end{bmatrix} \geq 0 \quad \forall t \geq t_k \tag{12}$$

To verify that this condition is satisfied, it is sufficient to check that, for each component of the error vector, assuming $e(t_0) \geq 0$,

$$\begin{aligned} & \text{if } \exists t^* \geq t_0 \text{ s.t. } e_i(t^*) = 0 \text{ and } e_j(t^*) \geq 0 \\ & (\forall j \neq i \in \{1, \ldots, 2n_x\}), \text{ then } \dot{e}_i(t^*) \geq 0 \end{aligned} \tag{13}$$

This means that every time a component e_i of the positive error vector vanishes, its derivative must be non negative in order to prevent the corresponding error from becoming negative.

The predictor design is based on the system model (1). The derivatives of the lower and upper bounds $(\dot{x}_P^-, \dot{x}_P^+)$ are computed by replacing parameters and variables in (1) by the bounds allowing to satisfy criteria (13).

Moreover, the predictor equations can be extended by introducing new variables resulting from state transformations.

$$\mathbf{z} = \mathbf{h}(\mathbf{x}, \mathbf{u}, \boldsymbol{\theta}) \tag{14}$$

The transformations have to be chosen in such a way that the evolution equations of the bounds can be simplified, and in particular, that terms involving uncertain parameters can be eliminated. Indeed, the growth rate of the interval sizes is amplified by the presence of these uncertain terms. From the equations for $\mathbf{z}$ and $\dot{\mathbf{z}}$, the predictor equations for $\mathbf{z}^-$ and $\mathbf{z}^+$ can be derived in a similar manner. Using the inverse transformation $\mathbf{h}^{-1}$, it is possible to combine intervals on the original variables $\mathbf{x}$, so as to obtain narrower prediction intervals.

If a confidence level is related to the intervals at time t_k, a lower bound of the integrity level can be derived for the intervals on the predicted state variables [4] and is given by

$$\begin{aligned} p\left(\mathbf{x}(t) \in [\mathbf{x}_P^-(t), \mathbf{x}_P^+(t)]\right) &= p\left(B_{\mathbf{x}(t)} = \text{true}\right) \\ &\geq \beta_{\mathbf{x}_k}\beta_{\boldsymbol{\theta}}\beta_{\mathbf{u}_k} \\ &\geq \beta_{obj} \end{aligned} \tag{15}$$

with $\beta_{\mathbf{x}_k}$, $\beta_{\boldsymbol{\theta}}$ and $\beta_{\mathbf{u}_k}$ the integrity related, respectively, to the state vector at t_k, the parameters and the inputs.

Indeed, the predictor is designed in such a way that it is sufficient for all the variables and parameters to be bounded at the present time to guarantee bounded intervals on state variables at future instants.

B. Correction step

At a measurement time, measurement intervals are combined by union and intersection operations. The resulting integrity is based on the Boolean random variable (6) related to each interval and their corresponding operations (8)-(11). Every possible combinations are considered such that the derived integrity satisfies the integrity objective.

The combination set is summarized in the following. Considering n intervals denoted $[a_1],\ldots,[a_n]$, let $\mathscr{C}^k$ define the set of k-combinations from the set of n elements $\{1,\ldots,n\}$. The number of elements in this set is given by :

$$n_c = \left(\begin{array}{c} n \\ k \end{array}\right) = \frac{n!}{k!(n-k)!}$$

The resulting intervals are provided by the following formulas

$$\mathbf{I}_h^k(i) = \bigcap_{j=\mathbf{i}_i(1)}^{\mathbf{i}_i(h)} \mathbf{U}^k(j) \quad k \in \{1,\ldots,n\} \tag{16}$$

$\mathscr{C}^k = \{\mathbf{c}_1,\ldots,\mathbf{c}_{n_c}\}$, describes all possible unions of k intervals among n, and consequently we have $\mathbf{U}^k(j) = [a_{\mathbf{c}_j(1)}] \cup \ldots \cup [a_{\mathbf{c}_j(k)}]$;

$h \in \{1,\ldots,h_{max}\}$ and $h_{max} = \left\lfloor \frac{n}{k} \right\rfloor$ is the maximum number of possible intersections between the intervals derived from the union operations;

$\mathscr{I}_h^k = \{\mathbf{i}_1,\ldots,\mathbf{i}_{n_i}\}$, describes all possible intersections of h intervals contained in $\mathbf{U}^k$ preventing that the original intervals appear twice.

Example : combination set for 4 intervals $[a_1]$, $[a_2]$, $[a_3]$, $[a_4]$.

- $k=1, h=1$: $[a_1]$, $[a_2]$, $[a_3]$, $[a_4]$;
- $k=1, h=2$: $[a_1]\cap[a_2]$, $[a_1]\cap[a_3]$, $[a_1]\cap[a_4]$, $[a_2]\cap[a_3]$, $[a_2]\cap[a_4]$, $[a_3]\cap[a_4]$;
- $k=1, h=3$: $[a_1]\cap[a_2]\cap[a_3]$, $[a_1]\cap[a_2]\cap[a_4]$, $[a_1]\cap[a_3]\cap[a_4]$, $[a_2]\cap[a_3]\cap[a_4]$;
- $k=1, h=4$: $[a_1]\cap[a_2]\cap[a_3]\cap[a_4]$;
- $k=2, h=1$: $[a_1]\cup[a_2]$, $[a_1]\cup[a_3]$, $[a_1]\cup[a_4]$, $[a_2]\cup[a_3]$, $[a_2]\cup[a_4]$, $[a_3]\cup[a_4]$;
- $k=2, h=2$: $([a_1]\cup[a_2])\cap([a_3]\cup[a_4])$, $([a_1]\cup[a_3])\cap([a_2]\cup[a_4])$, $([a_1]\cup[a_4])\cap([a_2]\cup[a_3])$;
- $k=3, h=1$: $[a_1]\cup[a_2]\cup[a_3]$, $[a_1]\cup[a_2]\cup[a_4]$, $[a_1]\cup[a_3]\cup[a_4]$, $[a_2]\cup[a_3]\cup[a_4]$;
- $k=4, h=1$:$[a_1]\cup[a_2]\cup[a_3]\cup[a_4]$.

Then, integrity is computed for each combination thanks to intersection and union formulas (8)-(11). Finally, the shortest non-empty interval satisfying the integrity objective is chosen.

C. Security measure

Besides integrity, it is interesting to quantify the probability of having a faulty estimation in a given time interval. To compute this probability, a binomial distribution is considered with the estimation process as a Bernouilli experiment. Indeed, at each measurement time, the measurement intervals are independent.

Definition 1: Binomial Distribution. Considering n independent experiments with a probability β of success. X, following a binomial distribution with parameters n and β, $X \sim B(n,\beta)$, has a probability mass function :

$$\phi(k;n,\beta) = \left(\begin{array}{c} n \\ k \end{array}\right) \beta^k (1-\beta)^{n-k} \tag{17}$$

which represents the probability of getting exactly k successes.

Property 1 : The probability of having one failure ($k = n-1$) is given by :

$$\phi(n-1;n,\beta) = n\beta^{n-1}(1-\beta) \tag{18}$$

If each experiment has its own probability of success /failure ($\beta_j/\alpha_j = 1-\beta_j,\ j \in N = \{1,\ldots,n\}$), the probability mass function related to one failure is given by :

$$\phi(n-1;n,\beta_1,\ldots,\beta_n) \quad = \quad \sum_{j=1}^{n} \left(\prod_{i \in N\setminus\{j\}} \beta_i \right) (1-\beta_j) \tag{19}$$

Property 2 : If $\beta_{min} = \min \beta_j \quad (j \in N = \{1,\ldots,n\})$ and $\beta_{min} \geq 1 - \frac{1}{n}$, then

$$\phi(n-1;n,\beta_1,\ldots,\beta_n) \leq n\beta_{min}^{n-1}(1-\beta_{min}) \tag{20}$$

This result can be proved recursively by searching monotonic domains of the function $\phi(n-1;n,\beta_1,\ldots,\beta_n)$.

Consequently, the probability of having one faulty estimation can be defined by :

Definition 2 : Security measure. Considering that n independent state vector estimations are carried out during a given time interval and having a minimum integrity level $\beta_{obj} \geq 1-1/n$ guaranteed for each estimation, the maximum probability of having a faulty estimation is given by :

$$\phi(n-1;n,\beta_{obj}) = n\beta_{obj}^{n-1}\left(1-\beta_{obj}\right) \tag{21}$$

Thus, it is possible to define an integrity objective related to the correction step such that it satisfies a given security measure expressing the probability of having a faulty estimation. Moreover, the predictor is useful to guarantee this integrity objective between two measurement times.

IV. Vehicle positioning : an autonomous underwater vehicle

In this section, the confidence interval observer is applied to an AUV positioning problem. To avoid handling a too complex model and without loss of generality, the vehicle is assumed moving in a horizontal plane and is described by a 3 degrees of freedom (DOF) model. The 3 DOF model considers 6 state variables grouped in two vectors, η and ν and two reference frames, the NED-frame and the BODY-frame (figure 1).

$$\eta = [x,y,\psi]^T \qquad \nu = [u,v,r]^T \tag{22}$$

where x and y are the vehicle origin location in the NED-frame, ψ is its heading, u and v are its linear velocity according the BODY-frame and r is the angular velocity with respect to the z-axis.

- **NED-frame**. The *North-East-Down* system x_n, y_n, z_n. It is usually defined as the tangent plane on the surface of the Earth and can be considered as fixed in a local area. For this system, x-axis points towards true North, the y-axis points towards the East while the z-axis points downwards normal to the Earth's surface.
- **BODY-frame**. This frame x_b, y_b, z_b is fixed to the vehicle and is thus a moving coordinate frame.

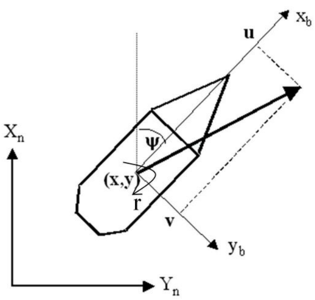

Fig. 1. Frames and 3 DOF vehicle state variables.

A. Vehicle modelling

Vessel's dynamics can be divided in two parts : *kinematics* which describes the velocity conversion from the BODY-frame to the NED-frame and *kinetics* which is the analysis of forces and momentums equilibrium.

The chosen evolution equations are given by

$$\begin{aligned} \dot{\eta} &= J(\eta)\nu \qquad (23) \\ M\dot{\nu} + C(\nu)\nu + D(\nu)\nu + g(\eta) &= \tau_c \qquad (24) \end{aligned}$$

$J(\eta)$ is the rotational matrix computing coordinates from the BODY-frame to the NED-frame.

$$J(\eta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\mathbf{M} \in \Re^{3\times 3}$ represents the system inertia matrix which includes added mass. $\mathbf{C}(\nu) \in \Re^{3\times 3}$ is the Coriolis-centripetal matrix (including added mass too). $\mathbf{D}(\nu) \in \Re^{3\times 3}$ is the damping matrix containing hydrodynamic damping. Generally, it contains linear and quadratic terms which are assumed uncoupled : $D(\nu) = D_L + D_Q(\nu)$. In this study, without loss of generality, only the linear term is considered. This means that the vehicle moves at a low speed. $\mathbf{g}(\eta) \in \Re^3$ is the vector of gravitational and buoyancy forces and moments. Moreover, this vector is assumed equal to zero by considering that the AUV is neutrally buoyant and that the centre of gravity and the centre of buoyancy are placed in the same vertical axis. Finally, $\tau_c \in \Re^3$ is the vector of control inputs. The nonlinear 3 DOF dynamic equation becomes :

$$\dot{\nu} = M^{-1}(\tau_c - C(\nu)\nu - D\nu) \qquad (25)$$

Nonlinear terms come from the Coriolis part of the equation. More information about marine vehicle modelling can be found in [3].

B. Sensor description

Several sensors can be used for AUV positioning, for instance:

- **Acoustic Positioning Systems**. Using beacons placed in the sea bed, they provide position (x, y) and heading (ψ) measurements. Because the acoustic signals are sent from and received by the AUV, noise sources come from the environment and a loss of signal can occur if obstacles are in the acoustic trajectory.
- **Inertial Navigation Systems**. Using accelerometers and gyros, these sensors measure the acceleration vector $(\dot{u}, \dot{v})$ and the velocity angle r with respect to the z-axis.
- **Radars**. They measure the frequency drift between an emitted signal and its reflection. The Doppler principle allows to derive velocity vector (u, v).

In this study, the main assumption is the availability of sensors providing intervals bounding the true state with some probability. So, the presented results can be extended to other vehicles (e.g other marine vessels or cars) described by similar dynamic equations and to other kind of sensors (e.g. GPS). Moreover, statistical information is assumed to be known such that confidence intervals can be derived. Finally, parameters are assumed given by guaranteed intervals and inputs are perfectly known.

C. Predictor description

First consider the evolution of the position vector η (23)

$$\begin{aligned} \dot{x} &= u\cos\theta - v\sin\theta \\ \dot{y} &= u\sin\theta + v\cos\theta \\ \dot{\theta} &= r \end{aligned}$$

The objective is to get prediction equations computing the evolution of the lower and the upper bounds related to x, y and θ. The corresponding equations are given by :

$$\begin{aligned} \dot{x}^+ &= (u\cos\theta)^+ - (\mathrm{v}\sin\theta)^- \\ \dot{y}^+ &= (u\sin\theta)^+ + (\mathrm{v}\cos\theta)^+ \\ \dot{\theta}^+ &= r^+ \\ \dot{x}^- &= (u\cos\theta)^- - (\mathrm{v}\sin\theta)^+ \\ \dot{y}^- &= (u\sin\theta)^- + (\mathrm{v}\cos\theta)^- \\ \dot{\theta}^- &= r^- \end{aligned} \qquad (26)$$

where $(.)^-$ and $(.)^+$ symbols denote lower and upper bounds of the elements inside the brackets.

Then consider the evolution equations of the velocity vector ν (24) written in the following form:

$$\begin{aligned} \dot{u} &= a + \alpha_u u + \alpha_{vr} vr + \alpha_{rr} r^2 \\ \dot{v} &= b + \beta_v v + \beta_r r + \beta_{uv} uv + \beta_{ur} ur \\ \dot{r} &= c + \gamma_v v + \gamma_r r + \gamma_{uv} uv + \gamma_{ur} ur \end{aligned} \qquad (27)$$

The predictor equations are derived in the same way which is presented here for u^+ and u^- :

$$\begin{aligned} \dot{u}^+ &= a^+ + \alpha_u^+ u^+ + \left(\alpha_{rr} r^2\right)^+ + (\alpha_{vr} vr)^+ \\ \dot{u}^- &= a^- + \alpha_u^- u^- + \left(\alpha_{rr} r^2\right)^- + (\alpha_{vr} vr)^- \end{aligned} \qquad (28)$$

In addition, state transformations can be used, which have the following forms :

$$z_{uv} = \gamma_{uv} v - \beta_{uv} r \qquad z_v = \gamma_v v - \beta_v r$$
$$z_{ur} = \gamma_{ur} v - \beta_{ur} r \qquad z_r = \gamma_r v - \beta_r r$$

Each of them allows to eliminate one term in the corresponding derivative, e.g.

$$\begin{aligned} \dot{z}_{uv} &= (\gamma_{uv} b - \beta_{uv} c) + (\gamma_{uv}\beta_v - \beta_{uv}\gamma_v)\, v + (\gamma_{uv}\beta_r - \beta_{uv}\gamma_r)\, r \\ &+ (\gamma_{uv}\beta_{ur} - \beta_{uv}\gamma_{ur})\, ur \\ &= k_{uv} + k_{uv}^{v} v + k_{uv}^{r} r + k_{uv}^{ur} ur \end{aligned}$$

This way, predictor equations become, e.g. for z_{uv}

$$\begin{aligned} \dot{z}_{uv}^{-} &= k_{uv}^{-} + (k_{uv}^{v} v)^{-} + (k_{uv}^{r} r)^{-} + (k_{uv}^{ur} ur)^{-} \\ \dot{z}_{uv}^{+} &= k_{uv}^{+} + (k_{uv}^{v} v)^{+} + (k_{uv}^{r} r)^{+} + (k_{uv}^{ur} ur)^{+} \end{aligned} \tag{29}$$

These equations have to be solved simultaneously with equations (26) and (28), which provide bounds on the original state variables. At each time, narrower intervals can be computed from relations between original and transformed variables in the following manner:

$$\begin{aligned} v^{-} &= max\left(v^{-},\, ((z_{uv} + \beta_{uv} r) / \gamma_{uv})^{-}\right) \\ v^{+} &= min\left(v^{+},\, ((z_{uv} + \beta_{uv} r) / \gamma_{uv})^{+}\right) \\ r^{-} &= max\left(r^{-},\, ((\gamma_{uv} v - z_{uv}) / \beta_{uv})^{-}\right) \\ r^{+} &= min\left(r^{+},\, ((\gamma_{uv} v - z_{uv}) / \beta_{uv})^{+}\right) \\ z_{uv}^{-} &= max\left(z_{uv}^{-},\, (\gamma_{uv} v)^{-} - (\beta_{uv} r)^{+}\right) \\ z_{uv}^{+} &= min\left(z_{uv}^{+},\, (\gamma_{uv} v)^{+} - (\beta_{uv} r)^{-}\right) \end{aligned} \tag{30}$$

These operations are similar for the other transformed variables z_{ur}, z_v and z_r. It is assumed that the initial intervals enclose the state variables so that the error vector $e(t_0) \geq 0$ (12). To sketch the demonstration, condition (13) is checked for a state variable from η and from ν.

- Upper bound on x : x^{+}

If $\exists t$ s.t. $x(t^*) = x^{+}(t^*)$ (other error components ≥ 0)

$$\begin{aligned} \dot{e}_x^{+}(t^*) &= \dot{x}^{+}(t^*) - \dot{x}(t^*) \\ &= \left((u\cos\theta)^{+} - (v\sin\theta)^{-}\right) - (ucos\theta - vsin\theta) \\ &= \left((u\cos\theta)^{+} - u\cos\theta\right) + \left(vsin\theta - (v\sin\theta)^{-}\right) \\ &\geq 0 \end{aligned}$$

- Upper bound on u : u^{+}

If $\exists t^*$ s.t. $u(t^*) = u^{+}(t^*)$ (other error components ≥ 0)

$$\begin{aligned} \dot{e}_u^{+}(t^*) &= \dot{u}^{+}(t^*) - \dot{u}(t^*) \\ &= \left(a^{+} + \alpha_u^{+} u^{+} + (\alpha_{rr} r^2)^{+} + (\alpha_{vr} vr)^{+}\right) \\ &- \left(a + \alpha_u u + \alpha_{vr} vr + \alpha_{rr} r^2\right) \\ &= (a^{+} - a) + (\alpha_u^{+} - \alpha_u)\, u + \left((\alpha_{rr} r^2)^{+} - \alpha_{rr} r^2\right) \\ &+ \left((\alpha_{vr} vr)^{+} - \alpha_{vr} vr\right) \geq 0 \end{aligned}$$

Similar conclusions can be drawn for the other state variables and the transformed variables z_{uv}, z_{ur}, z_v and z_r.

D. Simulated example

The confidence interval algorithm (section III) is now illustrated with the AUV positioning. Each state variable is measured by three sensors : 3 APS sensors for (x, y) and ψ, 3 gyros for r and 3 radars for (u, v). A common sampling time of $10s$ is used. Confidence intervals are computed by assuming Gaussian distribution. Information on the standard deviation can be found in Table I. Parameters are given by guaranteed intervals built with 10% error.

The integrity objective is $(1 - 10^{-8})$ and the integrity of each measurement interval is defined as $(1 - 10^{-4})$. In the correction step, the resulting integrity is computed for each state variable using (8) and (10) to reach the integrity objective by only using sensor intervals. The security measure (21) can be applied here because we have 360 measures in one hour ($T_s = 10s$) and the integrity objective is higher than the limit ($\beta_{obj} = 1 - 10^{-8} \geq 359/360$). The probability of having one bad estimation among the 360 estimation processes is given by

$$\begin{aligned} \phi(359; 360, \beta_{obj}) &= 360\left(1 - 10^{-8}\right)^{359} 10^{-8} \\ &\approx 3.6 \times 10^{-6} \end{aligned}$$

Concerning the prediction step, intervals generated by the predictor have to satisfy condition (15), which is, in this application,

$$p\left(\mathbf{x}(t) \in \left[\mathbf{x}_P^{-}(t), \mathbf{x}_P^{+}(t)\right]\right) \geq \beta_{\mathbf{x}_k} \tag{31}$$

At each measurement time t_k, β_{x_k} is computed thanks to (8) from intervals on state variables which will be used to reset the predictor. These intervals are computed like in the correction step by combination of unions and intersections operations (16). Moreover, the integrity objective has to be chosen such that condition (31) is satisfied. Two predictors are tested : the first one computes propagation of the whole state vector (26)-(28) and the second one computes in addition the evolution of bounds on variables z_{uv}, z_{ur}, z_v and z_r (29),(30).

x	y	ψ	u	v	r
10m	10m	0.2 rad	0.2 m/s	0.2 m/s	0.005 rad/s

TABLE I

STANDARD DEVIATIONS OF MEASUREMENT ERRORS.

Figure 2 shows interval state estimation when using the second predictor ((26),(28), (29) and (30)) and the integrity objective $(1 - 10^{-8})$. We first observe that the computed intervals enclose the true state variables as expected. The algorithm ensures that unknown state variables are bounded with a probability above $(1 - 10^{-8})$. Between two sampling times, the size of the predicted intervals usually grows fast due to the model uncertainty (e.g. for u and v velocities), especially with an increasing input command τ_c like in the simulation. However, frequent corrections allow the interval size to be periodically reduced. At all times, the integrity objective is therefore satisfied. Actually, at each measurement time, the computed integrity is $(1 - 10^{-8})$ resulting from the union of two intervals and between two measurement times, the resulting integrity is given by $\beta_{\mathbf{x}_k} \approx (1 - 6 \times 10^{-12})$ from the product of the integrity of each state variable. Indeed, the computed integrity of each state variable is $(1 - 10^{-12})$ (union of three intervals) so that $\beta_{\mathbf{x}_k}$ satisfies the integrity objective β_{obj}.

Finally, the interval sizes are compared for the two predictors (with and without the use of state transformations) in figure 3, clearly showing the improvement in the second configuration.

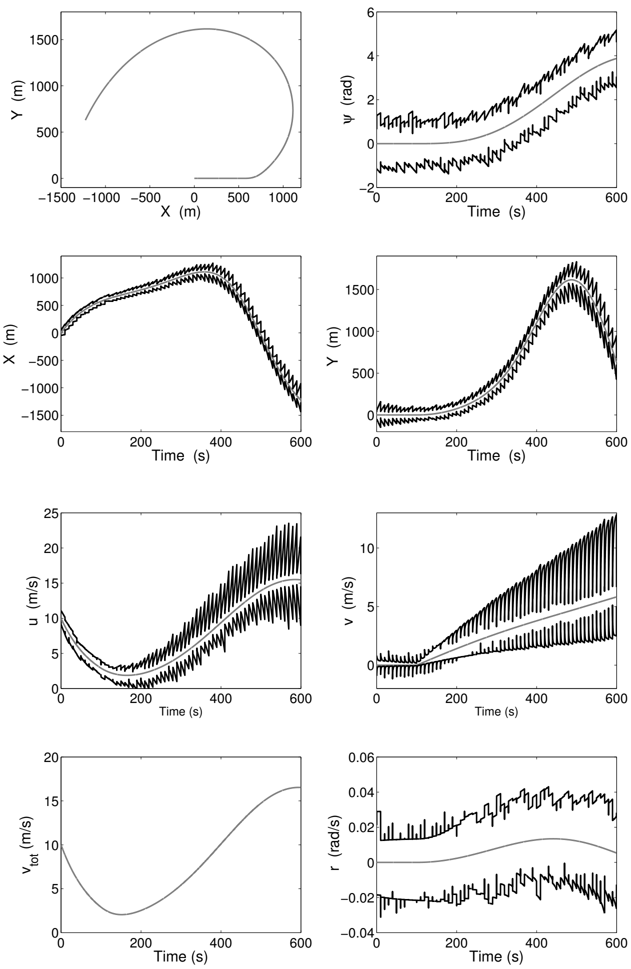

Fig. 2. Confidence interval state estimation : position x, y, heading ψ, linear (u, v) and angular r velocity.

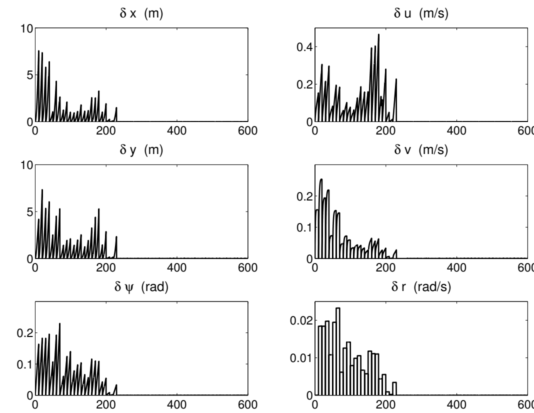

Fig. 3. Time evolution of the difference between the intervals size using predictor 1 (only (26),(28)) and predictor 2 ((26),(28), (29) and (30)).

V. Conclusion

In this work, a state estimation algorithm is developed which provides confidence intervals on the state variables. This algorithm is designed so as to always guarantee a specified integrity level by combining information (provided in the form of intervals) from a model and several sensors. The method proceeds in two steps, i.e. a predictor based on a vehicle description and a correction step at each sampling time combining measurement intervals by union and intersection operations.

In order to reduce interval size during the prediction step, state transformations are used in order to remove some penalizing terms in the vehicle model. The results show improvements (figure 3) which are however not very significant with regard to the increase in the computational load.

Moreover, a security measure is proposed based on the binomial distribution which represents the probability of having one faulty estimation during some period of time. Its definition is useful to estimate the reliability of safety-related process where the number of tolerated errors per hour is specified.

The developed method performs well and allows to satisfy safety demand in critical applications. This method is general and can be applied to various problems characterized by uncertainties on the initial state, inputs and parameters. Furthermore, it can handle any kind of statistical distribution.

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