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Continuous Path Following Control for Underactuated Systems With Bounded Actuation

Eric E. Sandoz and Petar V. Kokotović

***Abstract*— There exist many systems in our daily lives which are underactuated, subject to limits in actuation and which must navigate in constrained spatial environments. Furthermore, these systems can be described in uniform relative degree strict feedback form. Foremost among these are wheeled vehicles like passenger cars. Systems which navigate the physical environment often must follow precise geometric paths, but do not need to traverse the path at a specific speed profile. Hence we will utilize a path following approach, whereby path speed is considered an additional control degree of freedom.**

For such systems we present a control design method which will follow an arbitrary path with bounded error. The path following error itself is controllable. We then investigate conditions under which zero path following error is possible. Finally, we develop path speed limit conditions for a given path which guarantee zero path following error with control actions restricted to feasible values.

I. INTRODUCTION

We interact with many systems in our daily lives which are underactuated, subject to limits in actuation and which must navigate in constrained spatial environments. Foremost among these are wheeled vehicles like passenger cars, aircraft and watercraft. For reasons ranging from expense to limited human ability, most of these vehicles are underactuated. That is to say the number of independent degrees of freedom of control are less than the degrees of freedom which the driver/pilot must control. Traditional tracking controllers cannot be designed for such non-invertible systems. Moreover, most vehicles need only navigate geometric path (space) type problems, and not space-time problems simultaneously. A simple example is a passenger car parking maneuver. It is critical that the car control its position and orientation such that it ends up in the right spot, and avoids striking other vehicles. However, it is not critical that it perform this maneuver with precise timing, just that it completes the maneuver within a reasonable time period. In addition to being underactuated, all vehicles have finite bounds on control actuation, and most linear or non-linear controllers do not account for this explicitly. While both optimal controllers, Model Predictive Controllers (MPC) and Bang-Bang type controllers do explicitly address limits in actuation, they do so at the expense of computational complexity and difficult to realize discontinuous controls respectively.

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P.V. Kokotović is with Faculty of Electrical and Computer Engineering, University of California, Santa Barbara, USA petar@ece.ucsb.edu

E.E. Sandoz is a graduate student at University of California, Santa Barbara, USA, esandoz@ece.ucsb.edu

As work towards more autonomous vehicles proceeds, it is of interest that vehicles be capable of selecting their own paths, negotiating them all while remaining within their limits of actuation. It is also desirable to achieve this with a continuous controller which guarantees a level of robustness to model uncertainties and external disturbances.

To address these limitations, we present the following control method which produces a robust, computationally efficient, controller for underactuated systems in vectorial strict feedback form which decouples the path planning problem from the dynamic control problem and respects actuator limitations. The method is an extension of Skjetne et. al. [1] combined with concepts from Dacic et. al [2]. It utilizes vectorial backstepping, nonlinear damping, stability of interconnected systems and path following.

The control design presented here is applicable to systems in vectorial strict feedback form with uniform relative degree r or systems which, through a coordinate transformation, can be placed in this form.

$$\begin{aligned} \dot{\eta} &= \phi(\eta, \xi, u) \\ \dot{\xi}_1 &= f_1(\xi_1) + G_1(\xi_1)\xi_2 \\ \dot{\xi}_2 &= f_2(\bar{\xi}_2) + G_2(\bar{\xi}_2)\xi_3 \\ &\vdots \\ \dot{\xi}_r &= f_r(\xi) + G_r(\xi)u \\ y &= \xi_1 \\ & \xi_i \in \mathbb{R}^n,\ i = 1, \ldots, r,\ \bar{\xi}_i = (\xi_1^T\ \xi_2^T\ \cdots\ \xi_i^T)^T \end{aligned} \tag{1}$$

Where $\xi = \bar{\xi}_r$ is the stack of the ξ_i sate vectors, $y = \xi_1$ is the system's output, r is the system's uniform vector relative degree and $\phi(\eta, \xi, u)$ represent the system's zero dynamics, which are assumed to be locally input to state stable (ISS). The term $u \in \mathbf{U} \subset \mathbf{R}^m$ is the control. An underactuated system implies that the function $G_r(\xi) \in \mathbb{R}^{n\times m},\ m < n$ is not square. Thus the control law given in [1] is not directly implementable as it requires a square, nonsingular G_r.

While the design method presented in this paper can be applied without modification to systems with any finite relative degree, for simplicity of presentation and for focus on wheeled vehicles, we will use the following relative degree two rigid Newtonian dynamics which effectively describes the steering dynamics of a car.

$$\begin{aligned} \dot{\xi}_1 &= f_1(\xi_1) + G_1(\xi_1)\xi_2 \\ \dot{\xi}_2 &= f_2(\bar{\xi}_2) + G_2(\bar{\xi}_2)u \end{aligned} \tag{2}$$

Where $\xi_1 = (x\ y\ \psi)^T$ are the planar position and orientation variables in global coordinates and $\xi_2 = (v_u\ v_v\ v_\psi)^T$ are the longitudinal, lateral and angular velocities respectively in body coordinates. Three tire forces are accounted for, the longitudinal force F_u, the front tire lateral force F_{vf} and the rear tire lateral force F_{vr}. The control $u = (F_u \ F_{vf})^T$ is comprised of the longitudinal and lateral forces on the front tire in vehicle coordinates.

This paper is organized in the following sections: Section two presents the path following based navigation concept, explaining the advantages and formulation of a decoupled navigation and control problem. Followed by Geometric path following control design, where vectorial backstepping is employed to drive the state to follow a geometric path. This section includes the control design and error system analysis. The fourth section discusses the design of the path speed controller, which accomplishes a dynamic task. We present our conclusions in the last section.

II. Path Following Based Navigation Concept

Whether performed by a human or autonomously via a navigation algorithm, path planning to achieve a navigational goal requires information beyond the immediate state of the vehicle and its immediate surroundings. Average people in cars, athletes on the field and race car drivers all use sensors capable of scanning ahead of their current state, then choose a path which gets them closer to their next goal while avoiding obstacles. Moreover, each of the above examples constructs their paths as a sequence of practiced maneuvers. Passenger car drivers negotiate a right angle turn at an intersection by first picking a straight approach at an appropriate speed, then executing an arc followed by another straight line to exit the turn and continue on the road. Each aspect of each path segment was trained with an instructor. Similarly, a football (soccer) player has a finite set of moves or maneuvers. When playing, the athlete continually constructs a sequence of these moves to avoid an opponent and achieve higher level goals. The race car driver is the most explicit example as these athletes meticulously plan their *line* (i.e. path) through each curve in a course, and then for each lap adjust their car's actuators to stay on their *line*.

Given a geometric path, denoted $y_p(\theta(t))$, where $\theta \in \mathbf{R}_{\geq 0}$ is a scalar parameter defining the location along the path, the pilot/driver (i.e. controller) then continually adjusts the vehicle's actuators to keep the vehicle output $\xi_1(t)$ on the path. In our path following approach y_p is parameterized by scalar θ which itself can be controlled, thus gaining an extra degree of freedom for the control designer. The key difference between tracking, in which case the path $y_p(t)$ is an explicit function of time, and path following, is that with path following both the path position and speed can be controlled. For example, should a moving obstacle intersect $y_p(\theta_o)$, then the path velocity $\dot{\theta}$ can be brought to zero to wait for the obstacle to leave. Similar to a pedestrian walking in front of a car during a parking maneuver.

Another advantage of the path following dichotomy is that there is a strong correlation between required actuation force and path speed. For a car on the road the available cornering and/or tractive force is ultimately limited by the downforce of the vehicle and the coefficient of friction and tire slip. Thus a car cannot negotiate a 10m radius turn at an arbitrary velocity. Rather, it has a path speed limit determined by the tire force limits. For a given coefficient of friction, this path speed limit can be computed off-line for each maneuver. Thus a path planning algorithm need only choose path speeds below the speed limit for each segment, decoupling the path planning problem from the steering control problem.

Our control method employs the above dichotomy. We begin by assuming that the path y_p is given. This could come from an operator or a path planning algorithm. The path y_p is constructed from a sequence of feasible maneuvers. Each path element or maneuver $m_j(\cdot)$ is parameterized and the parameters belong to compact sets. The bounded parameter sets have been determined such that the execution of a maneuver will not demand more from the actuators than they can give.

$$\begin{aligned} y_p(\theta) &= \{y_{p,1}(\theta_1),\ y_{p,2}(\theta_2), \ldots,\ y_{p,N}(\theta_N)\} \\ y_{p,i} &\in M \end{aligned} \tag{3}$$

A given vehicle system has a maneuver set M consisting of a finite number of maneuvers which can be translated, rotated and scaled in sequence to form a continuous path. Each maneuver, like a turn or a swerve, can be parameterized by quantities like initial position, radius of curvature, and maneuver speed limit $\dot{\theta}_{max}$. It is reasonable to assume that our path is continuous, hence $y_{p,i}(\theta_{i,f}) = y_{p,i+1}(\theta_{i+1,0})$. For the rest of this paper we will focus on individual path segments or maneuvers and will thus drop the path segment indices.

III. Path Following Control Design

The path following controller consists of two parts, one accomplishes a path following task, discussed here, while the other controls velocity along the path, discussed in the next section. The first controller will drive the output to the path $y_p(\theta)$ given θ and its first two derivatives. The second controller drives the dynamics of $\theta(t)$ to ensure that forward motion along the path is guaranteed and actuation limits are respected.

We begin with the path following controller. We employ the Bicycle Model for steering dynamics [4].

Now let us look at the car steering problem as it fits the vectorial strict feedback form. (Rolling resistance is ignored).

$$\dot{\xi}_1 = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_u \\ v_v \\ v_\psi \end{bmatrix} \tag{4}$$

$$\dot{\xi}_2 = \begin{bmatrix} -C_d v_u^2 + 2 v_v v_\psi \\ -2 v_u v_\psi + c_1 F_{vr}(\xi_2) \\ c_3 F_{vr}(\xi_2) \end{bmatrix} + \begin{bmatrix} c_1 & 0 \\ 0 & c_1 \\ 0 & c_2 \end{bmatrix} \begin{bmatrix} F_u \\ F_{vf} \end{bmatrix} \tag{5}$$

$$c_1 = 2/m,\ c_2 = 2a/I,\ c_3 = -2b/I \tag{6}$$

Where m[kg] is the vehicle mass, a[m] is the distance from the center of gravity (CG) to the front axle, b[m] the distance from CG to rear axle, and I is the moment of inertia. A lumped parameter C_d contains the coefficient of aerodynamic drag and vehicle frontal cross-sectional area. Note that with respect to general rigid Newtonian systems in (2) there is no $f_1(\xi_1)$ term and G_2 is a constant matrix. These features will simplify our backstepping design.

Another important point is our choice of the forces F_u and F_{vf} as control variables. Of course wheel torque and wheel angle are the real control variables. We utilize the Pacejka tire model [7] which is a static nonlinear mapping between vehicle states and wheel angle to tire forces F_u and F_{vf}. For the region of normal operation, where the tire is not sliding, this function is invertible. Thus, provided our controller only commands forces achievable by the tire through a choice of wheel angle and torque, then we are free to work with whichever variable is most convenient. The last section of the paper addresses obeying these force limits.

$$\begin{aligned} \dot{\xi}_1 &= G_1(\xi_1)\xi_2 \\ \dot{\xi}_2 &= f_2(\xi_2) + G_2 u \end{aligned} \tag{7}$$

A. Backstepping control design

We begin by applying vectorial backstepping as in [1]. First, define our path following error z_1. Our control goal is to find a control law u such that $z_1(t) \to 0$ as $t \to \infty$.

$$\begin{aligned} z_1 &= \xi_1 - y_p(\theta) \\ \dot{z}_1 &= -\dot{y}_p(\theta) + G_1(\xi_1)\xi_2 \end{aligned} \tag{8}$$

From here forward we will drop the functional dependencies except where they are necessary for clarification. We observe that the state ξ_2 is affine in (8), which structurally looks like a control variable which we call a virtual control. Backstepping chooses a control law for ξ_2 called $\alpha_1(\xi_1, y_p)$ which would stabilize (8) if we could assign ξ_2 arbitrarily. We acknowledge that ξ_2 may not equal α_1 so we account for this virtual control error with the term $z_2 = \xi_2 - \alpha_1$. We will eventually choose our actual controller to drive z_2 to zero, which implies that ξ_2 is precisely a stabilizing control to $\dot{z}_1$ thus z_1 will converge to zero implying zero path following error.

$$\begin{aligned} z_2 &= \xi_2 - \alpha_1, \ (\xi_2 \ = \ z_2 + \alpha_1) \\ \dot{z}_1 &= -\dot{y}_p(\theta) + G_1(z_2 + \alpha_1) \\ \alpha_1 &= G_1^{-1}\left[A_1 z_1 + \dot{y}_p(\theta)\right] \\ \dot{z}_1 &= A_1 z_1 + G_1 z_2 \end{aligned} \tag{9}$$

Where A_1 is chosen as a Hurwitz matrix, and $\exists\ P_1 = P_1^T > 0,\ Q_1 = Q_1^T > 0$ such that $P_1 A_1 + A_1^T P_1 = -Q_1$. We choose a quadratic Lyapunov function as follows and examine its derivative

$$V_1 = z_1^T P_1 z_1 \tag{10}$$
$$\dot{V}_1 = 2 z_1^T P_1 (A_1 z_1 + G_1 z_2) \tag{11}$$
$$= -z_1^T Q_1 z_1 + 2 z_1^T P_1 G_1 z_2 \tag{12}$$

Observe that $\dot{V}_1$ is negative definite when $z_2 = 0$. Thus rendering $\dot{z}_1$ ISS with respect to the perturbation z_2. If we drive $z_2 \to 0$, then $z_1 \to 0$, and our output ξ_1 will converge to our path y_p.

Next, compute the derivative of z_2.

$$\begin{aligned} \dot{z}_1 &= A_1 z_1 + G_1 z_2 \\ \dot{z}_2 &= \dot{\xi}_2 - \dot{\alpha}_1 \\ &= f_2(\xi_2) - \dot{\alpha}_1 + G_2 u \end{aligned}$$

If we had a fully actuated system, and $G_2(\xi)$ was uniformly invertible, then we would be done by choosing $u = G_2^{-1}\left[A_2 z_2 - P_2^{-1} G_1^T P_1 z_1 - f_2(\xi) + \dot{\alpha}_1\right]$. And with Hurwitz $A_1,\ A_2$ our error system is exponentially stable. However, since our $G_2 \in \mathbf{R}^{3\times 2}$ is not square, the above controller is not an option. So let us go with the next best thing. Since, by assumption, G_2 has uniform full column rank, it has a uniformly invertible 2×2 sub-matrix which we will call G_{2p}, and a left over row called G_{2d}. The p and d subscripts denote *geometric path* and *dynamic path* respectively. Let us assume that G_2 is in the following form, or has been placed in this form by a permutation

$$G_2 = \begin{bmatrix} G_{2p} \\ G_{2d} \end{bmatrix} = \begin{bmatrix} S_p G_2 \\ S_d G_2 \end{bmatrix} \tag{13}$$

This partitioning can be accomplished by left multiplying G_2 by the following matrices $S_p,\ S_d$, which, when stacked, are the identity matrix. Furthermore, these partitioning operators can be used on the state z_2, the drift term $f_2(\xi)$, virtual control α_1 etc.

$$\begin{aligned} S_p &= [I_2\ \mathbf{0}]_{2\times 3},\ S_d \ = \ [\mathbf{0}\ 1]_{1\times 3},\ \begin{bmatrix} S_p \\ S_d \end{bmatrix}_{3\times 3} = I_3 \\ z_2 &= \begin{bmatrix} z_{2p} \\ z_{2d} \end{bmatrix} = \begin{bmatrix} S_p z_2 \\ S_d z_2 \end{bmatrix},\ \alpha_1 = \begin{bmatrix} \alpha_{1p} \\ \alpha_{1d} \end{bmatrix} = \begin{bmatrix} S_p \alpha_1 \\ S_d \alpha_1 \end{bmatrix},\ etc. \\ G_1 &= \begin{bmatrix} G_{1p} & G_{1d} \end{bmatrix} = \begin{bmatrix} G_1 S_p^T & G_1 S_d^T \end{bmatrix} \end{aligned} \tag{14}$$

Rewriting our system in its new partitioned format we have

$$\begin{aligned} \dot{\xi}_1 &= G_1(\xi_1)\xi_2 \\ \dot{\xi}_{2p} &= f_{2p}(\xi) + G_{2p} u \\ \dot{\xi}_{2d} &= f_{2d}(\xi) + G_{2d} u \end{aligned} \tag{15}$$

We now repeat the backstepping procedure above. The first step is the same, with the same α_1 and A_1 as in (9).

$$\dot{z}_1 = A_1 z_1 + G_{1p} z_{2p} + G_{1d} z_{2d} \tag{16}$$
$$\dot{z}_{2p} = f_{2p}(\xi) + G_{2p}(\xi) u - \dot{\alpha}_{1p} \tag{17}$$

At this point it is tempting to choose u just like we would have for the fully actuated system. Augmenting our Lyapunov function will help us see what is needed. Define

$$V_{2p} = V_1 + z_{2p}^T P_{2p} z_{2p} \tag{18}$$
$$\dot{V}_{2p} = \dot{V}_1 + 2 z_{2p}^T P_{2p} \dot{z}_{2p} \tag{19}$$
$$= -W_1 + 2 z_1^T P_1 (G_{1p} z_{2p} + G_{1d} z_{2d}) \tag{20}$$
$$+ 2 z_{2p}^T P_{2p} (f_{2p} + G_{2p} u - \dot{\alpha}_{1p}) \tag{21}$$

Where $W_1 = z_1^T Q_1 z_1$. We choose $u = u_p + u_d$ focused on the z_{2p} subsystem, but we leave an additional term u_d to deal with the z_{2d} subsystem, where the u_p component is

$$u_p = G_{2p}^{-1}\left[A_{2p} z_{2p} - f_{2p} - P_{2p}^{-1} G_{1p}^T P_1 z_1 + \dot{\alpha}_{1p}\right] \tag{22}$$

We now examine $\dot{V}_2$ with the application of $u = u_p + u_d$.

$$\dot{V}_2 = -W_2(z) + 2z_1^T P_1 G_{1d} z_{2d} + 2z_{2p}^T P_{2p} G_{2p} u_d \quad (23)$$

Where A_{2p} is Hurwitz and $\exists\ P_{2p} = P_{2p}^T > 0,\ Q_{2p} = Q_{2p}^T > 0$ such that $P_{2p}A_{2p} + A_{2p}^T P_{2p} = -Q_{2p}$, and $W_2(z) = z_1^T Q_1 z_1 + z_{2p}^T P_{2p} z_{2p} > 0,\ z \neq 0$. Finally we must augment our Lyapunov function one more time to account for the z_{2d} states, let $V_{2d} = V_2 + \frac{1}{2}z_{2d}^2$.

$$\begin{aligned}\dot{V}_{2d} =& -W_2(z) + z_{2d}(2G_{1d}^T P_1 z_1 + F_{2d}(z)) \\ &+ 2z_{2p}^T P_{2p} G_{2p} u_d + z_{2d} G_{2d} u_d\end{aligned} \quad (24)$$

Where $P_2 = \begin{bmatrix} P_{2p} & 0 \\ 0 & 1/2 \end{bmatrix}$. Once again we lack sufficient degree of freedom to cancel the indefinite term and make $\dot{V}_{2d}$ negative definite. The next best thing is to guarantee that $|z(t)| < \infty$ remain bounded. Our error system can now be written as

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_{2p} \\ \dot{z}_{2d} \end{bmatrix} = \begin{bmatrix} A_1 & G_{1p}(\xi_1) & G_{1d}(\xi_1) \\ -P_{2p}^{-1} G_{1p}(\xi_1)^T P_1 & A_2 p & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_{2p} \\ z_{2d} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ F_{2d}(z) \end{bmatrix} + \begin{bmatrix} 0 \\ G_{2p} \\ G_{2d} \end{bmatrix} u_d \quad (25)$$

$$\begin{aligned}F_{2d}(z) =& f_{2d}(\xi) - \dot{\alpha}_{1d} + \\ & G_{2d} G_{2p}^{-1}\left[A_{2p} z_{2p} - f_{2p} - P_{2p}^{-1} G_{1p}^T P_1 z_1 + \dot{\alpha}_{1p}\right]\end{aligned}$$

From this we see that choosing u_d to focus on the z_{2d} states will cause u_d to act as a disturbance to the z_{2p} subsystem. However, due to Hurwitz A_{2p}, $\dot{z}_{2p}$ is ISS and can tolerate some level of disturbance.

Let $\mu_i > 0$ be the smallest eigenvalue of each Q_i, and let us suppose that we have an upper bound on the indefinite term below for some domain of z, $2z_1^T P_1 G_{1d} z_{2d} + z_{2d} F_{2d}(z) \leq \kappa_{2d}\|z\|$.

$$\begin{aligned}\dot{V}_3 \leq& -\mu_1\|z_1\|^2 - \mu_{2p}\|z_{2p}\|^2 + \kappa_{2d}\|z\| + \\ & 2z_{2p}^T P_{2p} G_{2p} u_d + z_{2d} G_{2d} u_d\end{aligned} \quad (26)$$

Then we focus on the u_d term multiplied by z_{2d} and choose a suitable u_d.

$$u_d = -\kappa_d \frac{G_{2d}^T}{\|G_{2d}\|^2}\|z\| z_{2d} \quad (27)$$

Substituting this u_d into (26) gives

$$\begin{aligned}\dot{V}_{2d} \leq& -\mu_1\|z_1\|^2 - \mu_{2p}\|z_{2p}\|^2 + \kappa_{2d}\|z\| \\ & -2\kappa_d z_{2p}^T P_{2p} G_{2p} \frac{G_{2d}^T}{\|G_{2d}\|^2}\|z\| z_{2d} - \kappa_d\|z\| z_{2d}^2\end{aligned} \quad (28)$$

And since

$$-2\kappa_d z_{2p}^T P_{2p} G_{2p} \frac{G_{2d}^T}{\|G_{2d}\|^2}\|z\| z_{2d} \leq m_2\|z\|\|z_{2p}\||z_{2d}| \quad (29)$$

We remark that m_2 is controllable by the choice of P_{2p}, thus the bound can be made suitably small. So now our Lyapunov derivative is bounded by

$$\begin{aligned}\dot{V}_{2d} \leq& -\mu_1\|z_1\|^2 - \mu_{2p}\|z_{2p}\|^2 + \kappa_d(m_1 - z_{2d}^2)\|z\| \\ & + m_2\|z\|\|z_{2p}\||z_{2d}|\end{aligned} \quad (30)$$

Where $m_1 = \kappa_{2d}/\kappa_d$. Equation (30) shows that for $z_{2d}^2 > m_1$ the third term becomes negative and stabilizing. Choosing m_1 as small as possible is thus better which implies $\kappa_d >> \kappa_{2d}$. This will determine our error performance as $z = 0$ is an unstable equilibrium, but by choosing m_1 we can drive the error arbitrarily close to 0. The last term says that for large enough $\|z\|$ the system will always become unstable. This is due to the nonlinear damping term u_d overpowering the linear A_{2p} term as seen in (25). However, by choosing a small m_2 through manipulation of P_{2p} we can push this instability boundary out to any finite value.

Our underactuated control approach does not cause the tracking error to converge to zero, and it is not globally stabilizing. However, it can drive the tracking error arbitrarily close to zero, and the controller can be made to be stabilizing for any practical finite error. By virtue of our continuous control law, we also retain some level of robustness.

For arbitrary paths, our path following error z_1 will be bounded, but not zero on $z \in \mathbf{D}_z \subset \mathbf{R}^{2n}$. Where $\mathbf{D}_z$ is the domain of stable operation, and $0 \notin \mathbf{D}_z$. Physical intuition for a system can help us choose a meaningful, and not unnecessarily ambitious $\mathbf{D}_z$, as larger $\mathbf{D}_z$ comes at the expense of larger control actions. For the example of the car steering presented in the final section of this paper, we could reasonably restrict ourselves to path following errors of a few meters in position, and $\pm 60°$ for orientation. Then we design κ_d to just give us this without overpowering the system.

B. Making z_{2d} stable independent of u_d

While the design presented above generalizes to arbitrary relative degree systems, its principal drawback is the 0 element in the third column, third row of the matrix affine in z of equation (25). If this element could be made negative the z_{2d} state would be ISS. Furthermore, we then would have the chance for an asymptotically stable equilibrium. It turns out that the vehicle steering dynamics admits such a possibility. This comes from careful choice of A_1, which is a component of $\dot{\alpha}_1$. Since $\dot{\alpha}_1$ shows up in F_{2d} it follows that A_1 drives the $\dot{z}_{2d}$ dynamics.

We begin by rewriting $\dot{\alpha}_1$ to show z_{2d} explicitly. The ρ_z term is singled out as it is the one we will manipulate to stabilize z_{2d}.

$$\begin{aligned}\dot{\alpha}_1 &= \rho_0 + \rho_1 z_{2d} + \rho_z z_{2d} + \rho_2\ddot{\theta} \\ \rho_z &= G_1^{-1} A_1 G_{1d}\end{aligned} \quad (31)$$

Let us examine the z_{2d} dynamics more closely.

$$\begin{aligned}\dot{z}_{2d} &= c_3 F_{vr} - \dot{\alpha}_{1d} + G_{2d} G_{2p}^{-1}(A_{2p} z_{2p} - f_{2p} + \dot{\alpha}_{1p}) \\ v^T &= [0\ c_2/c_1]\ =\ G_{2d} G_{2p}^{-1} \\ \gamma &= c_3 F_{vr} - \rho_{0d} + v^T(A_{2p} z_{2p} - f_{2p} + \rho_{0p} + \rho_{1p} z_{2d}) \\ \dot{z}_{2d} &= \gamma + (v^T\rho_{zp} - \rho_{zd}) z_{2d} + (v^T\rho_{2p} - \rho_{2d})\ddot{\theta}\end{aligned} \quad (32)$$

If we can choose $(v^T\rho_{zp} - \rho_{zd}) < 0$, then we gain some stability margin for z_{2d}. The relevant terms are given below.

$$(v^T\rho_{zp} - \rho_{zd}) = \frac{c_2}{c_1}(-a_{1,3}\sin\psi + a_{2,3}\cos\psi) - a_{3,3} \quad (33)$$

From (33) we see that choosing $a_{1,3} = -r\frac{c_1}{c_2}\sin\psi$ and $a_{2,3} = r\frac{c_1}{c_2}\cos\psi$ where $r = -r_0 + a_{3,3},\ r_0 > 0$ retains A_1 as point-wise Hurwitz while making $(v^T\rho_{zp} - \rho_{zd}) = -r_0$. This is the desired effect.

$$A_1 = \begin{bmatrix} a_{1,1} & 0 & -r\frac{c_1}{c_2}\sin\psi \\ 0 & a_{2,2} & r\frac{c_1}{c_2}\cos\psi \\ 0 & 0 & a_{3,3} \end{bmatrix}$$

Note that since A_1 is diagonal with $a_{i,i} < 0$ it remains point-wise Hurwitz. While adding non-constant terms to A_1 changes $\dot{\alpha}_1$ as $\dot{A}_1 \neq 0$, fortunately, it does not change the dynamics of z_{2d}. It does change the z_{2p} dynamics, but these are taken care of by u_p. A careful calculation will show this.

Thus we now have

$$\dot{z}_{2d} = -r_0 z_{2d} + \gamma + (v^T\rho_{2p} - \rho_{2d})\ddot{\theta} \quad (34)$$

Finally we need to compute $P_1A_1(\xi_1)+A_1^T(\xi_1)P_1 = -Q_1$ to verify when Q_1 is positive definite. We will avoid additional complexity by trying to find a constant P_1. Start with diagonal.

$$Q_1 = \begin{bmatrix} 2p_{11}a_{11} & 0 & -p_{11}r_2\sin\psi \\ 0 & 2p_{22}a_{22} & p_{22}r_2\cos\psi \\ -p_{11}r_2\sin\psi & p_{22}r_2\cos\psi & 2p_{33}a_{33} \end{bmatrix} \quad (35)$$

Call $b = (-p_{11}r_2\sin\psi \ \ p_{22}r_2\cos\psi)^T$, where $r_2 = \frac{c_1}{c_2}(-r_0 + a_{33})$, then we can rewrite Q_1 as the sum of a diagonal matrix and a symmetric perturbation.

$$Q_1 = \begin{bmatrix} q_p & 0 \\ 0 & q_d \end{bmatrix} + \begin{bmatrix} 0 & b \\ b^T & 0 \end{bmatrix} \quad (36)$$

We know in terms of eigenvalue perturbations, the smallest (in magnitude) eigenvalue of the diagonal part of Q_1, called λ_1, will be perturbed at most by $\|b\|$, which is the largest eigenvalue of the perturbation matrix. Thus, we require that $\|b\| =< \lambda_1$. To simplify matters, we choose $p_{11} = p_{22}$ and we get the following requirement for $\|b\|$

$$(p_{11}^2r_2^2\sin\psi^2 + p_{22}^2r_2^2\cos\psi^2)^{\frac{1}{2}} = p_{11}r_2 \quad (37)$$

$$p_{11}r_2 \leq \min\{p_{11}|a_{11}|, p_{11}|a_{22}|, p_{33}|a_{33}|\} \quad (38)$$

The design goes like this. Pick $a_{33} < 0$ and $r_0 > 0$, then compute $r_2 = \frac{c_1}{c_2}(-r_0+a_{33})$. Pick p_{11} then choose $a_{11},\ a_{22}$ such that the above inequality is satisfied. Finally pick p_{33} large enough to satisfy the inequality.

Our final error system structure, after application of our nonlinear damping term is

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_{2p} \\ \dot{z}_{2d} \end{bmatrix} = \begin{bmatrix} A_1 & G_{1p}(\xi_1) & G_{1d}(\xi_1) \\ -P_{2p}^{-1}G_{1p}(\xi_1)^TP_1 & A_2p & 0 \\ 0 & 0 & -r_0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_{2p} \\ z_{2d} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \gamma + \rho_3\ddot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \Delta_{2p} \\ -1 \end{bmatrix} \kappa_d\|z\|z_{2d}$$

$$\rho_3 = (v^T\rho_{2p} - \rho_{2d}),\ \Delta_{2p} = -\frac{G_{2p}G_{2d}^T}{\|G_{2d}^T\|} \quad (39)$$

Remarks: (39) shows that if $\gamma + \rho_3\ddot{\theta} = 0$ then $z_{2d}(t) \to 0$. With $z_{2d} = 0$ since the upper 2×2 blocks of (39) are Hurwitz by construction, we have that $z_{2p}(t),\ z(t) \to 0$. This is instructive for designing our path parameter controller.

IV. Path Following Error Performance

In this section we will examine different ways of designing a controller for $\theta(t)$ to enhance performance, or mitigate problems. Recall that our path following control design from the previous section guarantees bounded path following error for $z \in \mathbf{D}_z$. We will first look at choosing path segments or maneuvers which admit zero path following error. Then we will look at designing a θ controller which can help reduce path following error. Finally, we will examine how careful design of the θ controller can be used to keep the control within its actuation limits.

A. Maneuver design for zero path following error

To achieve path following with zero error, we must render the point $z = 0$ an equilibrium, ideally a stable equilibrium. From (39) we see that all terms except $\gamma + \rho_3\ddot{\theta}$ are affine in z. Thus finding paths y_p such that $(\gamma+\rho_3\ddot{\theta})(z = 0) = 0$ will render $z = 0$ an equilibrium. Specifically we must determine the condition under which $\dot{z}_{2d}(z = 0) = 0$. First we must define some notation. Denote

$$y_p^\theta \triangleq \frac{\partial y_p}{\partial\theta},\ y_p^{\theta^m} \triangleq \frac{\partial^m y_p}{\partial\theta^m}$$

$$\dot{y}_p(\theta) = y_p^\theta\dot{\theta},\ \ddot{y}_p(\theta) = y_p^{\theta^2}\dot{\theta}^2 + y_p^\theta\ddot{\theta},\ \ldots \quad (40)$$

Zero path following error implies

$$\begin{aligned} z_1 &= 0,\ z_2 = 0 \\ \xi_1 &= y_p(\theta) \\ \xi_2 &= \alpha_1(z_1, y_p(\theta)) = G_1^{-1}\left[A_1z_1 + y_p^\theta\dot{\theta}\right] \end{aligned}$$

Substituting this into our expression for $\dot{z}_{2d}$

$$\dot{z}_{2d} = -(r_0 + \kappa_d\|z\|)z_{2d} + \gamma + \rho_3\ddot{\theta} \quad (41)$$

We see that the following choice for $\ddot{\theta}$ will render $z = 0$ a stable equilibrium and zero path following error has been achieved.

$$\ddot{\theta} = -\frac{\gamma}{\rho_3} \quad (42)$$

Of course there are no guarantees that this choice of $\ddot{\theta}$ will not become singular, or that θ will even go in the correct direction.

Whether or not this update law produces a meaningful path speed trajectory is a function of the path geometry itself. For example, requiring a car to move in a straight line but moving perpendicular to the direction of its wheels (something impossible except when sliding) will result in a trivial solution to $\ddot{\theta}$, that is $\dot{\theta} = 0$ is the only solution.

What is required is to define a target range for $\dot{\theta}$ and if (42) falls within this range then use it. Otherwise, default to another controller. For example

$$\ddot{\theta} = -\kappa_\theta d(\dot{\theta})$$

$$d(\dot{\theta}) = \begin{cases} \dot{\theta} - v_{min} & ,\ \dot{\theta} < v_{min} \\ \dot{\theta} - v_{max} & ,\ \dot{\theta} > v_{max} \\ \frac{-1}{\kappa_\theta}sat(-\frac{\gamma}{\rho_3}) & ,\ otherwise \end{cases} \quad (43)$$

where v_{min} and v_{max} are the minimum and maximum acceptable path velocities respectively, and the $sat(\cdot)$ function is chosen such that $d(\cdot)$ requests only feasible accelerations. This is one simplistic way to attempt zero path following error without designing something specific for a given path segment.

Another option for zero path following error is to consider the case of constant path speed ($\ddot{\theta} = 0$). Zero path following error dictates that we must find $\dot{\theta}$ such that $\gamma = 0$. We can rewrite $\gamma = \beta_0 + \beta_1\dot{\theta} + \beta_2\dot{\theta}^2$ to show the $\dot{\theta}$ dependence explicitly. Solving this implies finding $\dot{\theta} = cnst > 0$ such that $\gamma = 0$. This is not likely for most maneuvers, but it may work out for some.

One more way to approach this problem is to define a path speed profile, hence defining $\ddot{\theta}(t),\ \dot{\theta}(t),\ \theta(t)$, and then consider the path geometry which will guarantee zero path following error. In this approach we specify $y_p(\theta)$ for $n-1$ elements, and leave one element free. An example for a planar vehicle executing an arc turn would be

$$y_p(\theta) = \begin{bmatrix} x(\theta) \\ y(\theta) \\ \psi(\theta) \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ \psi(\theta) \end{bmatrix} \quad (44)$$

Let us suppose that $\psi(\theta) = q_1\theta + q_0$ is an affine function, and we will execute the arc at a constant path speed $\dot{\theta} = \dot{\theta}_0$. We must first compute the state trajectories under the assumption of $z = 0$.

$$\xi_1(q_1, q_0) = y_p,\ \xi_2(q_1, q_0) = G_1^{-1}(y_p) y_p^{\theta}\dot{\theta} \quad (45)$$

Substituting these functions (and the partial derivatives of y_p) into (39) and searching for the pair (q_1, q_0) which satisfy $\gamma + \rho_3\ddot{\theta} = 0$ will produce a path geometry which can be followed with zero error.

Heuristically this approach says that there is one path variable with which you have some flexibility, while the remaining $n-1$ path variables are not negotiable. In the above example we suggested that the heading angle ψ was flexible, while x and y were not.

B. Bounded Actuation

We now take a quick look at control effort, particularly as it pertains to path speed. Most practical actuators are limited in the force which they can produce, thus we are keenly interested in knowing which paths and for which speeds can zero path following error be had while not requiring control actions outside their physical limitations.

Looking at our control law assuming $z = 0$

$$\begin{aligned} & u(0, y_p, \dot{\theta}, \ddot{\theta}) = \\ & G_{2p}^{-1}\left[-f_{2p} + S_p(\rho_z + \rho_{1,1}\dot{\theta} + \rho_{1,2}\dot{\theta}^2 + \rho_{2,1}\ddot{\theta})\right] \end{aligned} \quad (46)$$

Where our virtual control law has been rewritten to show path dynamic variables $\dot{\alpha}_1 = \rho_z + \rho_{1,1}\dot{\theta} + \rho_{1,2}\dot{\theta}^2 + \rho_{2,1}\ddot{\theta}$. For some systems, like wheeled vehicles, G_{2p} is a constant matrix, thus it has a fixed norm. For this type of vehicle we see that

$$|u| \le d_z\|z\| + d_0 + d_1|\dot{\theta}| + d_{1,2}\dot{\theta}^2 + d_{2,1}|\ddot{\theta}| \quad (47)$$

$$\mu_{2p}|f_{2p} + S_p\rho_z| \le d_0,\ \mu_{2p}|S_p\rho_{i,j}| \le d_{i,j}$$

where $\mu_{2p} = \|G_{2p}^{-1}\|$. Since control effort is affine in $\dot{\theta},\ \dot{\theta}^2$ and $\ddot{\theta}$ it is clear that slower path speeds and accelerations imply lower control effort. Since the Pacejka tire function has force limits as a function of velocities $v_u,\ v_v,\ v_\psi$. Our actuation limits are defined by ξ_2 through $\dot{\theta}$ assuming zero error, whereas our commanded force is proportional to $\ddot{\theta}$. Thus, given $y_p(\theta)$, $\dot{\theta}$ determines the force at any point on the path, while $\ddot{\theta}$ more directly effects the commanded force. Then for a given path, we can establish a speed and acceleration limit (again something most likely given as a 2D feasibility set) such that staying within this set guarantees control actions below their saturation limit. Define such a function as $[\dot{\theta}_{max}\ \ddot{\theta}_{max}] = M(y_p, u_{max})$, where M is a path dependent speed limit function for bounded controls $u \in \mathbf{U}$.

V. Conclusions

A path following controller has been designed for systems in vectorial strict feedback form with uniform vector relative degree. The design was shown in detail for vehicle steering dynamics using the Bicycle Model. The longitudinal and lateral forces created by the front tire were considered as the control variables. With just two forces and three state variables to control the vehicle is underactuated. The controller guarantees bounded error trajectories for arbitrary paths provided the error state lies within a set $\mathbf{D}_z$ whose size is controllable. A redesign of the controller was presented which added an extra degree of stability to the z_{2d} subsystem making it possible to have a stable equilibrium in error coordinates. Finally an analysis of various methods for obtaining zero path following error with an underactuated vehicle was presented. Followed by a look at how the path following methodology can be used to ensure that actuator bounds are not exceeded by the controller.

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