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Planning of minimal-time trajectories for high speed autonomous vehicles

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ABSTRACT

The paper describes a method to compute the minimal-time trajectory for an autonomous vehicle driving at the force contact limit. It is proven that the optimal trajectory consists piecewise of logarithmic spirals and not of cothoids as is often thought. Simulation results are given for the interesting special case of a needle turn. These results are compared to normal polynomial interpolation.

1. INTRODUCTION

Most autonomous vehicles are not able to drive at speeds higher than 120 km/h. Thus, up to now the question how a high speed trajectory can be planned and executed has only been asked occasionally. In the context of driver assistance systems and autonomous high speed driving, this question becomes substantial. A possible answer could be helpful for efficient high speed driving, obstacle avoidance or stabilisation after an emergency [1, 2, 3]. The answer might also be of interest for line routeing or the design of racing tracks and speedways.

Minimal time trajectory planning for mobile robots has become a widespread field. One class of approaches uses a local description of state transitions in spatial- or configuration-space combined with exhaustive or intelligent search methods [e.g. 4, 5, 6, 7, 8]. Mostly holonomous systems are investigated. [9] employs exhaustive search. In [10] the search is optimized by a heuristics that is combined with an A* Algorithm. The local approach even with different optimisations is very computationally intensive, especially if high precision is required. Thus, flexible multi grid approaches have been investigated [11, 12, 13].

Variational methods have been applied to the question of minimal time trajectory for a very long period. For some special cases closed form solutions can be found [e.g. Brachiochrone], but with realistic constraints the resulting differential equations can be solved only numerically [e.g. 14, 15, 16]. This might be computationally expensive. Small changes in boundary conditions might heavily influence the solution and constraining conditions cannot always be included easily.

A third and widely used class of algorithms concatenates predefined curves. The different approaches propose the usage of circles, clothoids, polynoms of 3rd, 4th and 5th order (Bloss-curve) or sinoids [17]. In line routeing for high speed trains further curves are tested.

The advantage of piecewise approximation of the trajectory is the small number of parameters to be computed. Because we aim at a real-time suited method our approach will follow this last paradigm.

In the course of the paper the problem will be first defined in a more formal manner. The results for constant speed will be repeated briefly in Chapter 3. In Chapter 4 we will prove that the time-optimal curve for constant accelaration is a logarithmic spiral. Finally some simulation results will be illustrated and compared with polynomial trajectory computation in Chapter 5.

2. THE PROBLEM

We want to plan a trajectory from a point $\vec{x}_1 = \vec{x}_1(x_{1x}, x_{1y})$ at speed $\vec{v}_1 = \vec{v}_1(v_{1x}, v_{1y})$ to a point $\vec{x}_2 = \vec{x}_2(x_{2x}, x_{2y})$ at speed $\vec{v}_2 = \vec{v}_2(v_{2x}, v_{2y})$. The transfer should be performed in minimal time. We assume to be on a plane with constant friction coefficient value μ. The motion should fulfil several constraining conditions to be more realistic:

$v < v_{max}$	maximum speed is limited
$-a_{l\,max} < a_l < +a_{l\,max}$	longitudinal acceleration/breaking is limited
$-a_{n\,max} < a_n < a_{n\,max}$	normal acceleration is limited

Further the curvature $\rho = 1/r$ has an <u>upper</u> limit i.e. the vehicle is non holonomous and has a limit turning circle radius of r_{min}.

In this first approach the relation between longitudinal acceleration and speed caused by air resistance is neglected. Also the relation between longitudinal acceleration a_n and lateral acceleration a_l given by

$$a_{max} = \sqrt{a_1^2 + a_n^2} \qquad (1)$$

(Kamm's-circle), and tire dynamics like slip angle are ignored.

If we can compute this trajectory a complex path can be composed out of several of them. If further constraints have to be included e.g. a μ changing in space $\mu = \mu(x, y)$ this solution could be at least locally used where μ is approximately constant.

3. Minimal Time Trajectories For Constant Longitudinal Or Normal Acceleration

First we want to recall some common results for piecewise $a_l = const$ or $a_n = const$ along the path.

If start and target velocities are placed on a line the straight line with piecewise constant longitudinal acceleration results in the minimal-time transfer. Beside the minimal turning circle $r_{\min}$ another circle $r_{\max}$ exists, which is driven if the maximal normal force is reached. This circle limits the turning dynamic.

If there is at least one path between $\vec{x}_1$ and $\vec{x}_2$ which meets this requirements, a minimal time as well exists, to which several trajectories may correspond.

The function of minimal time $t_{\min}(\vec{x}_1, \vec{x}_2, \dot{\vec{x}}_1, \dot{\vec{x}}_2)$ is not steady for all starting conditions.

Example:

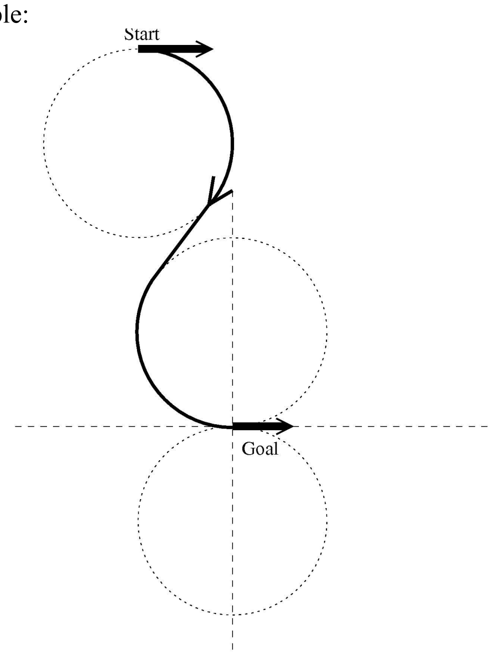

Fig. 1: Upper limit trajectory of start and goal have a common tangent

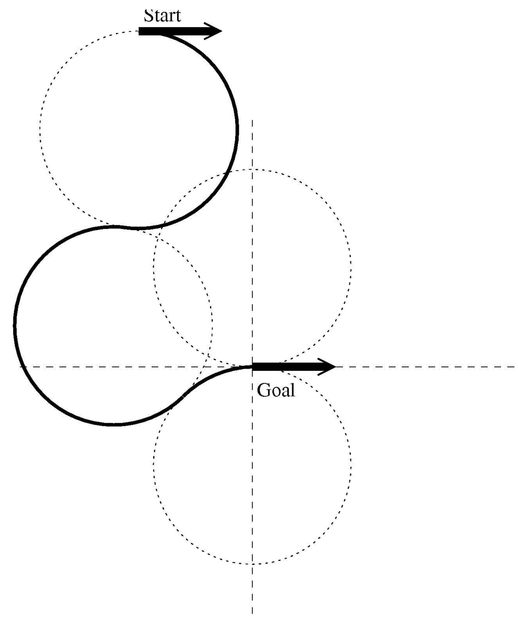

Fig. 2: Upper limit trajectory of start and goal have no common tangent

A slight change in start conditions obviously leads to a completely different solution (see Fig.1 and Fig. 2), even if large regions of $t_{\min}(\vec{x}_1, \vec{x}_2, \dot{\vec{x}}_1, \dot{\vec{x}}_2)$ are steady and differentiable.

Because of this it is not possible to find the minimal-time just by local planning. So, a combination of local and global planning is necessary in the general case.

Because this type of planning produces jumps along the path, often clothoids are fitted between regions of different ρ to reach a smooth run of the steering.

Special case: piecewise constant acceleration with

$$|\vec{v}_1| = |\vec{v}_2| \le v_{\max}$$

Acceleration takes only place on straight lines while turning is executed with a maximal positive $a_{n\max}$.

Three basic cases exist (Fig. 3) :

① Minimal turning circle $r_{\min}$ followed by a straight line

② Circle $r_{\min} \le r \le r_{\max}$ followed by a straight line

③ Limiting circle $r_{\max}$ (eventually followed by a straight line)

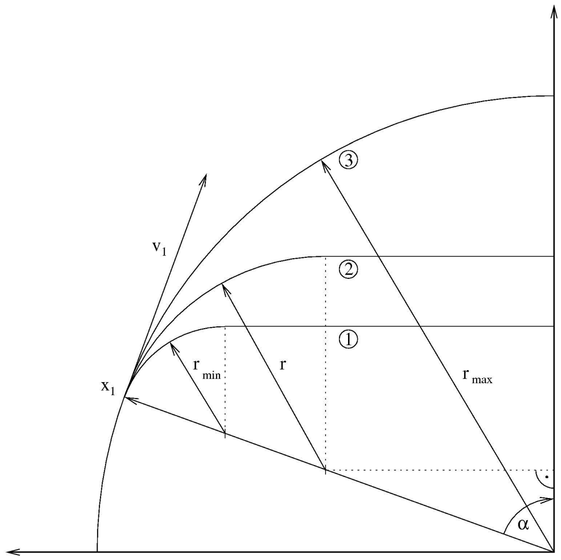

Fig. 3: cases of global planning with v_l = const

Path length of the given cases:

$$l(r) = l_c + l_s = r \cdot \alpha + \sin\alpha \cdot (r_{\max} - r) \tag{2}$$

with l_c: Length of circle section

l_s: Length of straight part

$l(r)$ is strictly monotonic increasing because:

$$\frac{dl(r)}{dr} = \alpha - \sin\alpha \geq 0 \tag{3}$$

for the complete interval $r_{\min} \leq r \leq r_{\max}$ and $0 \leq \alpha \leq 90°$.

Thus, ① is the time-optimal path for the above conditions. The time needed for the path is determined by the maximal normal acceleration $a_{n\max}$ and longitudinal acceleration $a_{l\max}$.

With: $r \cdot a_{n\max} = v_c^2$

$$t_c = \frac{l_c}{v_c} = \frac{r \cdot \alpha}{\sqrt{r \cdot a_{n\max}}} \tag{4}$$

$$l_s = \sin\alpha \cdot (r_{\max} - r) = \frac{1}{2} a_{l\max} \cdot t_s^2 + v_c \cdot t_s \tag{5}$$

$$t_s = \frac{1}{a_{l\max}} (-v_c \pm \sqrt{v_c^2 + 2\sin\alpha (r_{\max} - r)}) \tag{6}$$

$$t(r) = t_c + t_s \tag{7}$$

$$\frac{dt}{dr} = 0 \Rightarrow \text{Minimum} \tag{8}$$

This leads to a form, for which the solution for $t_{\min}$ depends on α, $a_{l\max}$, $a_{n\max}$ and $v_{\max}$. Thus, all of the cases above can possibly be the minimal-time trajectory, depending on the actual parameter set.

4. Minimal Time Trajectories Generated By Longitudinal And Normal Acceleration

The same (as Fig. 3) important special cases exist, assuming that we are starting at (0, 0) with $\vec{v} = (v_0, 0)$ (in local coordinates) and that the curvature changes of the trajectories are monotonous.

The upper limit trajectory is driven by $a_{l\max}$ and $a_{n\max}$ and the lower limit trajectory by $a_{l\min}$ and $a_{n\min}$. Below a certain speed v_{ls} only the non holonomy of the vehicle determines its behaviour in the turning circle. Drifting is not allowed.

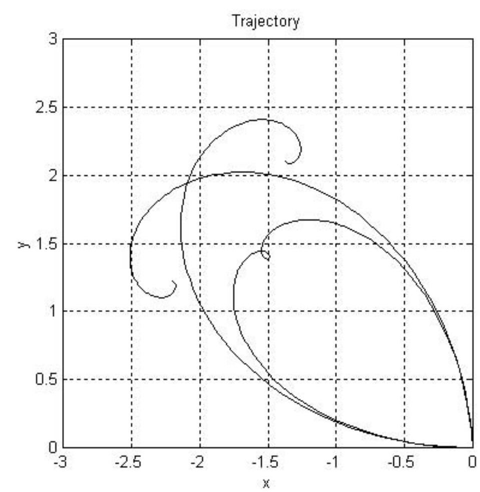

Fig. 4: cases for arbitrary acceleration

Analogously to Fig. 3 three cases exist for the starting condition:

1.) Upper limit trajectory of start and goal have a common tangent with the right direction (see case 1 Fig. 3). In this case the minimal time will be reached by accelerating along the upper limit tangent and part of the tangent of the start point and decelerating along the rest of the tangent and upper limit tangent of the goal.
2.) The upper limit trajectories of start and goal have no appropriate tangent but the lower limit trajectories have (see case 2 Fig. 3). This is the case of a steep turn which will be investigated in the sequel.

3.) There is no appropriate tangent either for the lower limit trajectories. This will lead to a detour (as in Fig. 2). We will not further investigate this case here.

For an optimal turn the vehicle must drive the minimal curvature radius $\rho_{\min}$, which is determined by $a_{l\max}$.

$$F_{\max} = \mathrm{m} \cdot a_{\max} \leq \mathrm{m} \cdot \mathrm{g} \cdot \mu_H \tag{9}$$

$$a_{\max} \leq \mathrm{g} \cdot \mu_H \tag{10}$$

By neglecting banking of the curve:

$$\dot{v}_l = \frac{dv_l}{dt} << a_n \Rightarrow a_{\max} = a_{n\max} \tag{11}$$

Otherwise the vehicle will drift. It is not clear if it is possible to minimize further the time to drive the curve by drifting, but a lot of clues from motor sports indicate that only without drift minimal times can be reached. In this paper we will consider only this case.

With v_l (t) being the longitudinal speed in direction of the tangent of the curve (i.e. in local coordinates):

Curvature: $$\rho\,(\mathrm{t}) = v_l^2(t) \cdot \frac{m}{F_n} \tag{12}$$

Because of driving at the limit normal acceleration:

$$\rho(t) = v_l^2(t) \cdot \frac{m}{F_{n\max}} = v_l^2(t) \cdot c_1 \tag{13}$$

The speed is given by:

$$v_l\,(\mathrm{t}) = \frac{ds(t)}{dt} \tag{14}$$

Thus

$$\frac{d\rho(t)}{dt} = \frac{d(v_l^2(t))}{dt} \cdot c_1 = 2 \cdot v_l(t) \cdot \frac{dv_l(t)}{dt} \cdot c_1 \tag{15}$$

With (14)

$$dt = \frac{d\rho(t)}{2 \cdot c_1 \cdot v_l(t) \cdot \dot{v}_l(t)} = \frac{ds(t)}{v_l(t)} \tag{16}$$

By rearranging:

$$\frac{d\rho}{ds} = \frac{2 \cdot c_1 \cdot v_l(t) \cdot \dot{v}_l(t)}{v_l(t)} = 2 \cdot c_1 \dot{v}_l(t) \tag{17}$$

The shortest time will be reached if

$$\dot{v}_l(t) = a_{l\max} = const. \tag{18}$$

because the vehicle is driving at contact limit. The result is

$$\boxed{\frac{d\rho}{ds} = 2 \cdot a_{n\max} \cdot a_{l\max} = const.} \tag{19}$$

This is the characteristic equation of a logarithmic spiral

$$r = a \cdot e^{k \cdot \varphi} \tag{20}$$

with r: Radius parameter
a: Base radius parameter
k: Angle parameter
φ: Angle

For the logarithmic spiral:

$$s(r) = \frac{1}{k}\sqrt{1+k^2} \cdot r \tag{21}$$

with $s(r)$: Arc length

$$\rho(r) = \sqrt{1+k^2} \cdot r \tag{22}$$

with $\rho(r)$: Curvature

Thus

$$\frac{ds(r)}{dr} = \frac{1}{k}\sqrt{1+k^2} \tag{23}$$

$$\frac{d\rho(r)}{dr} = \sqrt{1+k^2} \tag{24}$$

With (23):

$$dr = \frac{ds(r)}{\frac{1}{k}\sqrt{1+k^2}} = \frac{d\rho(r)}{\sqrt{1+k^2}} \tag{25}$$

By rearranging:

$$\frac{d\rho}{ds} = \frac{\sqrt{1+k^2}}{\frac{1}{k}\sqrt{1+k^2}} = k \tag{26}$$

These ratios could also be used for the local description of the general case

$$a_{\max} = \sqrt{a_{l\max}^2 + a_{n\max}^2} = const \tag{27}$$

Because locally it implies after inserting (19) into (27) and considering (26):

$$\begin{aligned} &2a_{l\max} \cdot \sqrt{a_{\max}^2 - a_{l\max}^2} \\ &= 2\dot{v}_{l\max} \cdot \sqrt{a_{\max}^2 - \dot{v}_{l\max}^2} = const = k \end{aligned} \tag{28}$$

The characteristic parameter of the curve can be locally determined directly by the speed.

$$r = \exp(2\dot{v}_{l\max}\sqrt{a_{\max}^2 - \dot{v}_{l\max}^2}) \tag{29}$$

In the case of maximum speed change the resulting path is a logarithmic spiral. The positions that can be reached in equal time (isotime -line) are placed on a cycloid.

To understand the behaviour of the logarithmic spiral, speed and transition time for local coordinates were computed.

ρ_1 is given by v_1 at the starting point x_1. It is interesting to note, that for a given v_1 and $a_{n\max}$ the spiral is unambiguously given. I.e. the parameter a has no influence. Thus, it can be set to 1. This also means that one goal point can only be reached with a certain direction at minimal time.

To compare this solution with clothoides which are utilised in most approaches it can be said: clothoids are intended for turning the steering steadily at constant velocity v_l. This is most convenient for a human driver. But an automatic system does not rely on simple handling or simple control functions. The proposed manoeuvre is optimised for minimal time and not for driving comfort i.e by exhausting the acceleration limits of the vehicle.

5. Examples And Comparison With Hemitesch' Interpolation

To illustrate the results of the preceding chapter, we computed an example of a turning manoeuvre with the starting point $\vec{x}_1 = \vec{x}_1(0\,,\,0)$ at speed $\vec{v}_1 = \vec{v}_1(-\sqrt{3}\,,\,0)$ to a point $\vec{x}_2 = \vec{x}_2(-1\,,\,2)$ at speed $\vec{v}_2 = \vec{v}_2(\sqrt{1.9}\,,\,0)$ with $a_{\max} = 1$.

The given example was selected of the second type mentioned in chapter 3. Because of the invariance to parameter a, two logarithmic spirals are necessary. The computed osculation point of the two spirals has an identic velocity vector. This point was found with the method of iterative gradient descend. It is not clear if such a point exits for all configurations, but it was found in all our numerical experiments. The resulting trajectory is shown in Fig. 5.

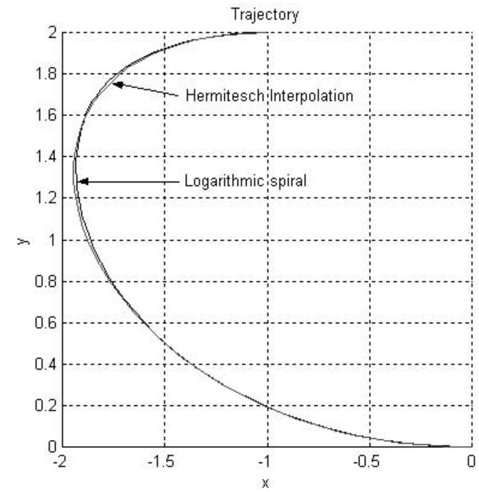

Fig. 5: minimal-time trajectory with logarithmic spirals and with Hermitesch Interpolation of 3rd order

Velocity and curvature are steady along the trajectory. Longitudinal and normal acceleration are steady and have a jump in their first derivative at the meeting point. The acceleration to this point is $a_1 = -0.43$ and after the meeting point $a_2 = -0.89$. The constraints are fulfilled and the course will be passed in minimal time. Additionally a circle has to be fitted between the two spirals if a given acceleration constraint for $a_{n\max}$ has to be met.

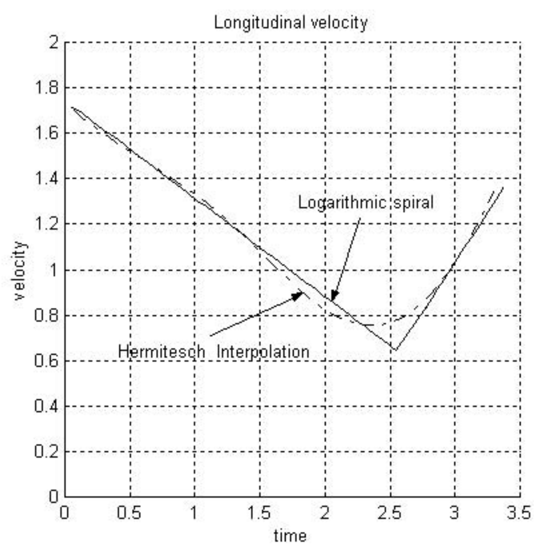

Fig. 6: Longitudinal speed over time

To compare these results with a conventional numerical 3rd order method, Hermitesch Interpolation was employed. The same boundary conditions and contraints were applied. With the polynom

$$P(t) = c_3 \cdot t^3 + c_2 \cdot t^2 + c_1 \cdot t^1 + c_0 \qquad (30)$$

with c_i being 2-dimensional vectors:

$$c_3 = (-1.8\,,\,0.4)$$
$$c_2 = (1.6\,,\,-2.8)$$
$$c_1 = (1.2\,,\,2.4)$$
$$c_0 = (0\,,\,0)$$

We get the solution also shown in fig. 5.

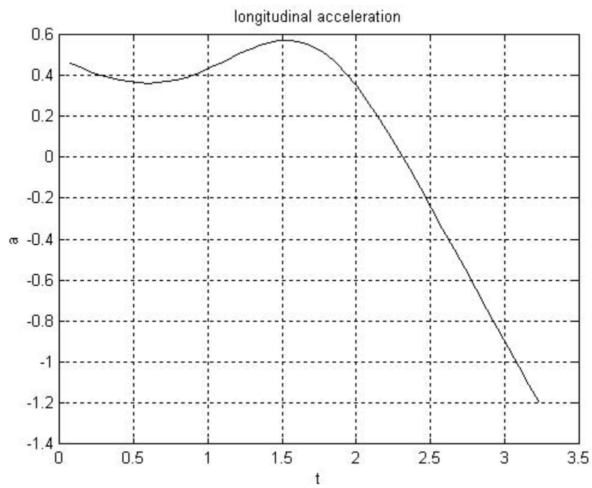

Fig. 7: Longitudinal acceleration of Hermitesch Interpolation

The resulting time t_{min} = 3,35 is the same (with 1 % numerical approximation errors) for the logarithmic spiral and the Hermitesch Interpolation (see fig. 6), but the Hermitesch Interpolation is exceeding the accelerations limits (see fig. 7).

6. CONCLUSION

We propose a new approach for planning extreme trajectories for high speed autonomous driving. It is shown that minimal time trajectories piecewise consist of logarithmic spirals (and circles, respective straight lines, which are special cases of the spiral). An example how to compute such a manoeuvre is shown and compared to a polynomial approach. Of course the actual model contains a lot of simplifications. E.g. the finite change time of normal acceleration should be included as well as the connection between normal and longitudinal acceleration, especially at limit trajectories. Most probably such influences can only be integrated in the model by numerical methods which will be investigated next.

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