9. Appendix I

All equations preceding Eq. 14 are referred to from the main draft. We will use the following identity in the below derivations and we use $d = N_w$.

$$\int_{\mathbf{x}} e^{-\frac{1}{2}\mathbf{x}^T A\mathbf{x}+\mathbf{b}^T\mathbf{x}} \mathrm{d}^n\mathbf{x} = \sqrt{\frac{(2\pi)^n}{det(A)}} e^{\frac{1}{2}\mathbf{b}^T A^{-1}\mathbf{b}} \tag{14}$$

$\mathrm{d}^n\mathbf{x}$ is the n-dimensional volume differential.

9.1. Sampling distributions

Let $d = N_w$.

$$\begin{aligned}
p(b_m|\mathbf{\Theta}\backslash\{a_m, b_m\},\mathbf{r}) &= \frac{\int_{a_m} p(\mathbf{\Theta},\mathbf{r})\mathrm{d}a_m}{\int_{a_m}\sum_{b_m} p(\mathbf{\Theta},\mathbf{r})\mathrm{d}a_m} \\
&= \frac{\int_{a_m} p(\mathbf{r}|\mathbf{\Theta})p(\mathbf{\Theta})\mathrm{d}a_m}{\int_{a_m}\sum_{b_m} p(\mathbf{r}|\mathbf{\Theta})p(\mathbf{\Theta})\mathrm{d}a_m} \\
&= \frac{\int_{a_m} p(\mathbf{r}|\mathbf{\Theta})\prod_k p(b_k)p(a_k|b_k)p(\sigma^2)p(\mathbf{c})\mathrm{d}a_m}{\int_{a_m}\sum_{b_m} p(\mathbf{r}|\mathbf{\Theta})\prod_k p(b_k)p(a_k|b_k)p(\sigma^2)p(\mathbf{c})\mathrm{d}a_m}
\end{aligned}$$

canceling terms that are independent of a_m and b_m and compute $p(b_m = 1|\mathbf{\Theta}\backslash\{a_m, b_m\}, \mathbf{r})$

$$p(b_m = 1|\mathbf{\Theta}\backslash\{a_m, b_m\}, \mathbf{r}) = \frac{\int_{a_m} p(\mathbf{r}|\mathbf{\Theta})p(a_m|b_m = 1)p(b_m = 1)\mathrm{d}a_m}{\int_{a_m}\sum_{b_m} p(\mathbf{r}|\mathbf{\Theta})p(a_m|b_m)p(b_m)\mathrm{d}a_m} \tag{15}$$

$$= \frac{\int_{a_m} p(\mathbf{r}|\mathbf{\Theta})p(a_m|b_m = 1)p(b_m = 1)\mathrm{d}a_m}{\int_{a_m} p(\mathbf{r}|\mathbf{\Theta})p(a_m|b_m = 1)p(b_m = 1)\mathrm{d}a_m + \int_{a_m} p(\mathbf{r}|\mathbf{\Theta})p(a_m|b_m = 0)p(b_m = 0)\mathrm{d}a_m} \tag{16}$$

$$p(b_m = 1|\mathbf{\Theta}\backslash\{a_m, b_m\}, \mathbf{r}) = \frac{NUM_1}{NUM_1 + DEN_1} \tag{17}$$

where

$$NUM_1 = \int_{a_m} p(\mathbf{r}|\mathbf{\Theta})p(a_m|b_m = 1)p(b_m = 1)\mathrm{d}a_m \tag{18}$$

Let $A = diag(\mathbf{a})$, where $diag(\mathbf{a})$ indicates the constructing diagonal matrix with the elements from the vector $\mathbf{a}$. From eq. 4 and eq. 2.

$$NUM_1 = \lambda \int_{a_m} \frac{1}{\sqrt{(2\pi(\sigma^2)^d)}} e^{\frac{(\mathbf{r}-A\mathbf{b})^T(\mathbf{r}-A\mathbf{b})}{2\sigma^2}} \frac{1}{(2\pi)^{\frac{1}{2}}\sigma_a} e^{-\frac{a_m^2}{2\sigma_a^2}} \mathrm{d}a_m$$

substituting $b_m = 1$ and ignoring the terms that are independent of a_m and b_m.

$$= \frac{\lambda C_1}{(2\pi)^{\frac{d+1}{2}}(\sigma^2)^{\frac{d}{2}}\sigma_a} \int_{a_m} e^{-\left[-\frac{1}{2}d_m a_m + \frac{1}{2}\left(\frac{1}{\sigma_a^2} + \frac{1}{\sigma^2}\right)a_m^2\right]}\mathrm{d}a_m$$

where $f = \{\frac{1}{\sigma^2} + \frac{1}{\sigma_a^2}\}$ and C_1 is constant. From eq. 14,

$$= \frac{\lambda C_1}{(2\pi)^{\frac{d+1}{2}}\sigma_a(\sigma^2)^{\frac{d}{2}}} \sqrt{\frac{2\pi}{f}} e^{\left(\frac{d_m^2}{2f}\right)} \tag{19}$$

$b_m = 0$, DEN_1 become as follows

$$= \frac{C_1(1-\lambda)}{(2\pi)^{\frac{d+1}{2}}(\sigma^2)^{\frac{d}{2}}\sigma_a} \int_{a_m} e^{-\left(\frac{a_m^2}{2\sigma_a^2}\right)}\mathrm{d}a_m \tag{20}$$

$$= \frac{C_1(1-\lambda)}{(2\pi)^{\frac{d}{2}}(\sigma^2)^{\frac{d}{2}}}. \tag{21}$$

Using eq. 19 and 21, eq. 17 can be written as

$$p(b_m = 1|\mathbf{\Theta}\backslash\{a_m, b_m\}, \mathbf{r}) = \frac{\lambda_{1,m}}{\lambda_{1,m} + 1 - \lambda} \tag{22}$$

where

$$\lambda_{1,m} = \frac{\lambda}{\sigma_a}\sqrt{\frac{1}{f}} e^{\left(\frac{d_m^2}{2f}\right)}$$

$$\begin{aligned}
p(a_m|\mathbf{\Theta} \setminus \{a_m\}, \mathbf{r}) &= \frac{p(\mathbf{\Theta}, \mathbf{r})}{\int_{a_m} p(\mathbf{\Theta}, \mathbf{r})\mathrm{d}a_m} = \frac{p(\mathbf{r}|\mathbf{\Theta})p(\mathbf{\Theta})}{\int_{a_m} p(\mathbf{r}|\mathbf{\Theta})p(\mathbf{\Theta})\mathrm{d}a_m} \\
&= \frac{p(\mathbf{r}|\mathbf{\Theta})\prod_k p(b_k)p(a_k|b_k)p(\sigma^2)p(c)}{\int_{a_m} p(\mathbf{r}|\mathbf{\Theta})\prod_k p(b_k)p(a_k|b_k)p(\sigma^2)p(c)\mathrm{d}a_m}
\end{aligned}$$

canceling the terms independent of a_m.

$$= \frac{p(\mathbf{r}|\mathbf{\Theta})p(a_m|b_m)}{\int_{a_m} p(\mathbf{r}|\mathbf{\Theta})p(a_m|b_m)\mathrm{d}a_m} = \frac{NUM_2}{DEN_2}$$

from eq. 18, $DEN_2 = \frac{NUM_1}{p(b_m=1)}$

$$p(a_m|\mathbf{\Theta} \setminus \{a_m\}, \mathbf{r}) = \frac{\frac{1}{(2\pi)^{\frac{d+1}{2}}\sigma_a(\sigma^2)^{\frac{d}{2}}} e^{\frac{1}{2}\left[2d_m a_m - \left(\frac{1}{\sigma_a^2} + \frac{1}{\sigma^2}\right)a_m^2\right]}}{\frac{1}{(2\pi)^{\frac{d+1}{2}}\sigma_a(\sigma^2)^{\frac{d}{2}}}\sqrt{\frac{2\pi}{f}} e^{\left(\frac{d_m^2}{8f}\right)}} \tag{23}$$

by completing the square.

$$\sqrt{\frac{f}{2\pi}} e^{\frac{f}{2}\left[\frac{1}{f}d_m a_m - a_m^2 - \frac{(d_m)^2}{4f^2}\right]} \tag{24}$$

$$= \mathcal{N}\left(\frac{d_m}{f}, \frac{1}{f}\right) \tag{25}$$

$$\begin{aligned}
p(\sigma^2|\mathbf{\Theta}\backslash\{\sigma^2\}, \mathbf{r}) &= \frac{p(\mathbf{r}, \mathbf{\Theta})}{\int_{\sigma^2} p(\mathbf{r}, \mathbf{\Theta})\mathrm{d}\sigma^2} = \frac{p(\mathbf{r}|\mathbf{\Theta})p(\mathbf{\Theta})}{\int_{\sigma^2} p(\mathbf{r}|\mathbf{\Theta})p(\mathbf{\Theta})\mathrm{d}\sigma^2} \\
&= \frac{p(\mathbf{r}|\mathbf{\Theta})\prod_k p(b_k)p(a_k|b_k)p(\sigma^2)p(c)}{\int_{\sigma^2} p(\mathbf{r}|\mathbf{\Theta})\prod_k p(b_k)p(a_k|b_k)p(\sigma^2)p(c)\mathrm{d}\sigma^2}
\end{aligned}$$

cancelling the terms that are independent of the σ^2

$$= \frac{p(\mathbf{r}|\mathbf{\Theta})p(\sigma^2)}{\int_{\sigma^2} p(\mathbf{r}|\mathbf{\Theta})p(\sigma^2)\mathrm{d}\sigma^2}$$

$$= \frac{\frac{\beta^\alpha(\sigma^2)^{-\alpha-1-\frac{d}{2}}}{\Gamma(\alpha)(2\pi)^{\frac{d}{2}}} e^{-\frac{\mathbf{d_3}^T\mathbf{d_3}}{2\sigma^2} - \frac{\beta}{\sigma^2}}}{\frac{\beta^\alpha}{\Gamma(\alpha)(2\pi)^{\frac{d}{2}}}\int_{\sigma^2}(\sigma^2)^{-(\alpha+\frac{d}{2})-1} e^{-\frac{\frac{1}{2}(\mathbf{d_3}^T\mathbf{d_3})+\beta}{\sigma^2}}\mathrm{d}\sigma^2}$$

Let $\mathbf{d_3} = \mathbf{r} - A\mathbf{b}$

(26)

From the definition of the $\mathcal{IG}$ distribution, the denominator can be expressed as

$$\frac{\beta^\alpha}{\Gamma(\alpha)(2\pi)^{\frac{d}{2}}}\int_{\sigma^2}(\sigma^2)^{-(\alpha+\frac{d}{2})-1} e^{-\frac{\frac{1}{2}(\mathbf{d_3}^T\mathbf{d_3})+\beta}{\sigma^2}}\mathrm{d}\sigma^2$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)(2\pi)^{\frac{d}{2}}}\frac{\Gamma(\alpha+\frac{d}{2})}{\left\{\frac{1}{2}(\mathbf{d_3}^T\mathbf{d_3})+\beta\right\}^{\alpha+\frac{d}{2}}}.$$

Substituting in eq. 26,

$$
\begin{aligned}
p(\sigma^2|\mathbf{\Theta}\backslash\{\sigma^2\},\mathbf{r}) &= \frac{\frac{\beta^{\alpha}\left(\sigma^2\right)^{-\alpha-1-\frac{d}{2}}}{\Gamma(\alpha)(2\pi)^{\frac{d}{2}}} e^{-\frac{\mathbf{d_3}^T\mathbf{d_3}}{2\sigma^2}-\frac{\beta}{\sigma^2}}}{\frac{\beta^{\alpha}}{\Gamma(\alpha)(2\pi)^{\frac{d}{2}}}\frac{\Gamma(\alpha+\frac{d}{2})}{\left\{\frac{1}{2}(\mathbf{d_3}^T\mathbf{d_3})-\beta\right\}^{\alpha+\frac{d}{2}}}} \\
&= \frac{\left\{\frac{1}{2}(\mathbf{d_3}^T\mathbf{d_3})-\beta\right\}^{\alpha+\frac{d}{2}}\left(\sigma^2\right)^{-\left(\alpha+\frac{d}{2}\right)-1} e^{-\frac{\frac{1}{2}\mathbf{d_3}^T\mathbf{d_3}+\beta}{2\sigma^2}}}{\Gamma(\alpha+\frac{d}{2})} \\
&= \mathcal{IG}(\alpha+\frac{d}{2},\frac{1}{2}\mathbf{d_3}^T\mathbf{d_3}+\beta)
\end{aligned}
$$