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FRAME MULTIPLIER AND IRREGULAR GABOR FILTERS WITH APPLICATION IN TIME FREQUENCY MASKING

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Abstract

This paper presents an overview and a summary on the mathematical background for time-frequency masking filters based on the concept of 'frame multipliers'. Gabor multipliers are a current topic of research, known in signal processing as Gabor filters. Frame multipliers are a generalization of this type of time variant filters to frames without further structure. After analysis, before synthesis the coefficients are multiplied by a fixed pattern, the so called symbol. The dependency of the operator on the symbol and frames is presented here. The paper also investigates irregular Gabor frames. In particular the results on irregular Gabor filters are given such as the continuous dependency of Gabor filters on the symbol, the lattice and the windows. An approximation of arbitrary matrices by irregular Gabor multipliers is presented. Focusing on applications the finite-dimensional discrete case becomes important. An algorithm is presented on how to iteratively invert the Gabor frame operator (for regular lattices) numerically efficient by using 'double preconditioning'. Finally a concept is presented how to implement a filter, which approximates the simultaneous and temporal masking known in psychoacoustics. As the linear frequency scale (in Hz) is not applicable to human perception, the Bark scale was introduced, consequently this filtering can be seen as an irregular Gabor multiplier with adaptable mask. The current paper reports on work in progress and presents some representative results only. An outlook on further results is given.

INTRODUCTION

The relevance of signal processing in today's life is evident, for example in the shape of DSL or MP3. If theory and application, mathematics and engineering, work together, coherent results can be reached and a high synergy effect can be observed. This has been demonstrated in wavelet theory [8] for example. Although the Fourier transformation and the Short Time Fourier Transformation are used for quite some time, only in the last couple of years, a application-oriented scientific field combining mathematics and engineering grew. This connection in the *Gabor theory* [14] has lead to many interesting results. This work should be seen right at this connection.

In many applications a modification is used on the coefficients of the analysis, e.g. with Fourier analysis [10]. In the last couple of years algorithms have been investigated, that does not use time-invariant, but time-variant filtering [20]. The so called *Gabor multipliers* [13] are a subkind of time-variant filters. Instead of factors fixed for every spectrum, a fixed map for the whole time-frequency plane is used, with which the time-frequency coefficients are multiplied. To connect theory and application, algorithms are needed. The continuous, infinite theory is not fitted for this goal, so the theory has to be developed for the discrete, finite dimensional case.

One technology which is used heavily in everyday life is MP3, the MPEG1 Layer 3 coding [21]. This is used to reduced the digital size of sound signal. There a special coder is used, which uses a model for the human audio perception. It is known in psychoacoustics, that not all parts of an audio signal can be perceived by a human. Some components mask other parts near to them in time or frequency. Clearly filtering out this data will result in reducing the data size without any subjective quality losses. An idea of how to extend known masking algorithms to a time-frequency model is given in the end of this work. This is done by using a Gabor multiplier. As the linear frequency scale Hz is not very well-fitted to the auditory perception, another frequency sampling chosen. This leads to irregular Gabor multipliers. This work starts with the investigation of an even more general case, the *frame multipliers*.

This work gives a summary of [2] by presenting representative results.

FRAME MULTIPLIERS

A sequence $\Psi = (\psi_k | k \in K) \subseteq \mathcal{H}$ is called a *frame* [6] for the Hilbert space $\mathcal{H}$, if constants $A, B > 0$ exist, such that

$$A \cdot \|f\|_{\mathcal{H}}^2 \leq \sum_k |\langle f, \psi_k \rangle|^2 \leq B \cdot \|f\|_{\mathcal{H}}^2 \ \forall \ f \in \mathcal{H} \tag{1}$$

It is call a *Bessel sequence* if only the right inequality is required to be fulfilled.

In the following let $\Psi = (\psi_k)_{k \in K}$ be a frame in $\mathcal{H}_1$ with frame bounds A,B and $\Phi = \{\phi_k\}_{k \in K}$ in $\mathcal{H}_2$ with bounds A',B'. For a bounded sequence $m \in l^\infty(K)$ let the *frame multiplier* be the operator $\mathbf{M}_{m,\Phi,\Psi} : \mathcal{H}_1 \to \mathcal{H}_2$, defined by

$$\mathbf{M}(f) := \mathbf{M}_{m,\Phi,\Psi}(f) = \sum_k m_k \langle f, \psi_k \rangle \phi_k. \tag{2}$$

Theorem 1 *1. Given a sequence $m \in l^\infty$ $\mathbf{M}$ is a well defined bounded operator with $\|\mathbf{M}\|_{Op} \leq \sqrt{B'}\sqrt{B} \cdot \|m\|_\infty$. Furthermore the sum $\sum_k m_k \langle f, \psi_k \rangle \phi_k$ converges unconditionally for all $f \in \mathcal{H}_1$.*

2. $\left(\mathbf{M}_{m,(\phi_k),(\psi_k)}\right)^* = \mathbf{M}_{\overline{m},(\psi_k),(\phi_k)}$. Therefore if m is real-valued and $\phi_k = \psi_k$, $\mathbf{M}$ is self-adjoint.

3. If the sequence m converges to 0, i.e. $m \in c_0$, $\mathbf{M}$ is a compact operator.

4. If $m \in l^1$, $\mathbf{M}$ is a trace class operator with $\|M\|_{trace} \leq \sqrt{B'}\sqrt{B}\ \|m\|_1$. And $tr(M) = \sum_k m_k \langle \phi_k, \psi_k \rangle$.

5. If $m \in l^2$, $\mathbf{M}$ is a Hilbert Schmidt operator with $\|M\|_{\mathcal{HS}} \leq \sqrt{B'}\sqrt{B}\ \|m\|_2$.

For the definition of the used operator classes refer e.g. to [7].

Theorem 2 *The operator $\mathbf{M}$ depends continuously on m, (ψ_k) and (ϕ_k), in the following sense: Let $(\psi_k^{(l)})$ and $(\phi_k^{(l)})$ be sequences indexed by $l \in \mathbb{N}$.*

1. Let $m^{(l)} \to m$ in l^1, $(\psi_k^{(l)})$ and $(\phi_k^{(l)})$ be frames with upper bounds $B_1^{(l)}$ and $B_2^{(l)}$, such that there exists $\mathbf{B_1}$ and $\mathbf{B_2}$ with $B_1^{(l)} \leq \mathbf{B_1}$ and $B_2^{(l)} \leq \mathbf{B_2}$. Let the sequences $(\psi_k^{(l)})$ and $(\phi_k^{(l)})$ converge uniformly to (ψ_k) respectively (ϕ_k). Then $\left\| M_{m^{(l)},(\psi_k^{(l)}),(\phi_k^{(l)})} - M_{m,(\psi_k),(\phi_k)} \right\|_{trace} \to 0$ for $l \to \infty$.

2. Let $m^{(l)} \to m$ in l^2 and let the frames $(\psi_k^{(l)})$ respectively $(\phi_k^{(l)})$ converge to (ψ_k) respectively (ϕ_k) in an l^2 sense, i.e. $\forall \varepsilon > 0\ \exists N$ such that $\sqrt{\sum_k \left\| \psi_k^{(l)} - \psi_k \right\|_{\mathcal{H}}^2} < \varepsilon$ for all $l \geq N$. Then $\left\| M_{m^{(l)},(\psi_k^{(l)}),(\phi_k^{(l)})} - M_{m,(\psi_k),(\phi_k)} \right\|_{\mathcal{HS}} \to 0$ for $l \to \infty$.

3. Let $m^{(l)} \to m$ in l^∞ and let the frames $(\psi_k^{(l)})$ respectively $(\phi_k^{(l)})$ converge to (ψ_k) respectively (ϕ_k) in an l^1 sense, i.e. $\forall \varepsilon > 0\ \exists N$ such that $\sum_k \left\| \psi_k^{(l)} - \psi_k \right\|_{\mathcal{H}} < \varepsilon$ for all $l \geq N$. Then $\left\| M_{m^{(l)},(\psi_k^{(l)}),(\phi_k^{(l)})} - M_{m,\psi_k,\phi_k} \right\|_{Op} \to 0$ for $l \to \infty$.

For more on this topic also refer to [3].

GABOR FRAMES AND FILTERS

For $\lambda = (\tau, \omega)$ define the time-frequency shift operator $\pi(\lambda) = M_\omega T_\tau$ with the *modulation* $(M_\omega f)(t) = e^{2\pi i \omega t} f(t)$ and *translation* $(T_\tau f)(t) = f(t-\tau)$. Let $g \in L^2(\mathbb{R})$ be a non zero function. Let Λ be a countable subset of $\mathbb{R}^2$, called the *(irregular) lattice*. The set of time-frequency shifts

$$\mathcal{G}(g, \Lambda) = \{\pi(\lambda) g : \lambda \in \Lambda\} \tag{3}$$

is called an (irregular) *Gabor system* [18, 19]. If it is a frame, it is called (irregular) *Gabor frame*. The analysis operator is identical to the *Short Time Fourier Transformation* [1] : $\mathcal{V}_g(f)(\tau, \omega) := \langle f, M_\omega T_\tau g \rangle$.

For the definition of the Wiener amalgam spaces $W(B, l^p)$ used in the following refer to [17]. The class $S_0(\mathbb{R})$ is the so-called Feichtinger's algebra [16] defined by:

$$S_0(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) \,\middle|\, \mathcal{V}_f f \in L^1(\mathbb{R}^2) \right\} \tag{4}$$

A lattice Λ is called *separated*, if there exists a $\delta > 0$ such that $|\lambda - \lambda'| > \delta$ for all $\lambda \neq \lambda' \in \Lambda$. It is called *relatively separated*, if it is a finite union of separated lattices.

Proposition 3 *Let Λ be a relatively separated lattice $\subseteq \mathbb{R}^2$. Then for a $g \in S_0$ the system (g, Λ) forms a Bessel sequence in $L^2(\mathbb{R})$.*

For a function $m : \mathbb{R}^{2d} \to \mathbb{C}$, *Gabor filters* or *Gabor multipliers* are frame multipliers for Gabor frames:

$$\mathbf{G}_{m,\gamma,g}(h) = \sum_{\lambda \in \Lambda} m(\lambda) \langle f, \pi(\lambda) g \rangle \pi(\lambda) \gamma. \tag{5}$$

There are many ways to incorporate time-variant filters. Gabor filters have a lot of advantages, see [20], to be easily and efficiently implementable is one of them.

We call two lattices $\Lambda = \{\lambda_k\}$, $\Lambda' = \{\lambda'_k\}$ with a common index set K δ-*similar*, if $|\lambda_k - \lambda'_k| \leq \delta \quad \forall k \in K$, denoted by $\mathfrak{s}(\Lambda, \Lambda') \leq \delta$.

Theorem 4 *Let $g, \gamma \in W(C_0, l^\infty)$, let Λ be a relatively separated irregular lattice, such that (g, Λ) (γ, Λ) form a pair of Bessel sequences for $L^2(\mathbb{R})$. Let $m \in W(C_0, l^1)$, then the trace-class operator $\mathbf{G}_{m,g,\gamma}$ depends continuously on m, g, γ and Λ, in the following sense: Let $g^{(l)} \to g$, $\gamma^{(l)} \to \gamma$ in $W(C_0, l^\infty)$. Let $\Lambda^{(\delta)}$ be lattices such that $\mathfrak{s}\left(\Lambda, \Lambda^{(\delta)}\right) \leq \delta$. Let $m^{(l)} \to m$ in $W(C_0, l^1)$. Then*

$$\mathbf{G}_{m^{(l)}, g^{(l)}, \gamma^{(l)}, \Lambda^{(\delta)}} \to \mathbf{G}_{m,g,\gamma,\Lambda} \textit{ in the trace class}$$

for $\delta \to 0$, $l \to \infty$.

Theorem 5 *Let $g, \gamma \in S_0(\mathbb{R}^d)$, let Λ be a δ-separated irregular lattice. Let $m \in W(C_0, l^2)$, then the Hilbert Schmidt operator $\mathbf{G}_{m,g,\gamma}$ depends continuously on m, g, γ and Λ, in the following sense: Let $g^{(l)} \to g$, $\gamma^{(l)} \to \gamma$ in $S_0(\mathbb{R}^d)$. Let $\Lambda^{(\delta)}$ be lattices such that $\mathfrak{s}\left(\Lambda, \Lambda^{(\delta)}\right) \leq \delta$. Let $m^{(l)} \to m$ in $W(C_0, l^2)$. Then*

$$\mathbf{G}_{m^{(l)},g^{(l)},\gamma^{(l)},\Lambda^{(\delta)}} \to \mathbf{G}_{m,g,\gamma,\Lambda} \text{ in } \mathcal{H}S$$

for $\delta \to 0$, $l \to \infty$.

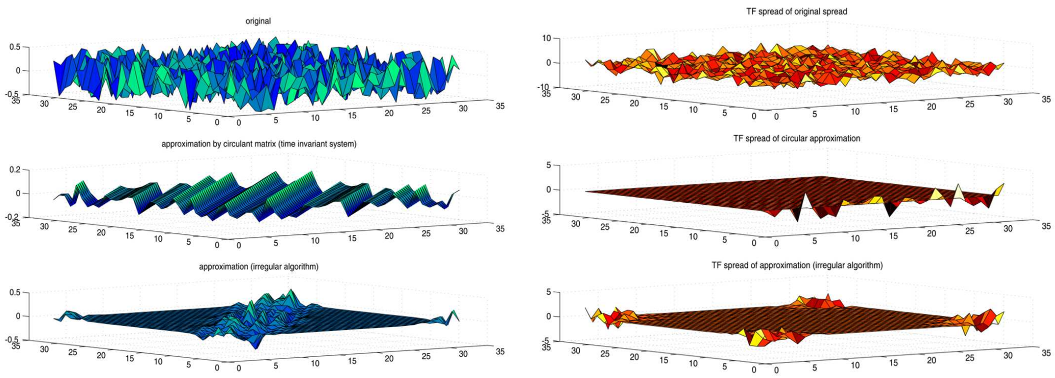

Figure 1: *(Left:) Comparison of random system matrix and the best approximations by time-invariant and Gabor filter. (Right:) Time-Frequency spread of these matrices.*

More application-oriented, an algorithm to find the best approximation of arbitrary system matrices by irregular Gabor filters was implemented (compare to [12]). For a random system matrix in Figure 1 a comparison of the best approximation by a time-invariant and a Gabor filter is displayed, both comparing the matrices themselves and also their time-frequency spread. Note that, as expected, these two different way of filtering show different time-frequency behavior.

DOUBLE PRECONDITIONING FOR GABOR FRAMES

Applications and algorithms work with finite dimensional data, where questions of numerical efficiency and stability arise. For an analysis-synthesis system an important property is *perfect resynthesis*. By general frame theory we know that for regular Gabor frames, i.e. $\Lambda = \{(la, kb)\,|k, l \in \mathbb{Z}\}$ for the *lattice parameters* a,b, this property can be guaranteed, if the dual window is used for synthesis. This is found by applying the inverse of the Gabor frame operator S [14] on the original window. In the finite dimensional case $\mathbb{C}^L$ the Gabor frame operator S has a very special, well-known sparse structure [22]. This can be used to formulate a new iterative algorithm for the inversion of S by *double preconditioning*. Both the projection on diagonal matrices, $S \mapsto D(S)$, and on the circulant matrices, $S \mapsto C(S)$, is used to find a preconditioning matrix P to speed up the convergence of the iterative algorithm.

$$P = C\left(D(S)^{-1} \cdot S\right)^{-1} D(S)^{-1} \tag{6}$$

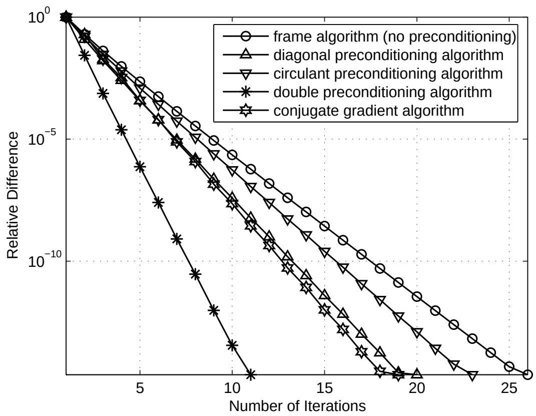

Figure 2: *Calculation of canonical dual: Convergence in iterations, relative difference of iteration steps (Gaussian window, $n = 1440$, $a = 32$ and $b = 30$.)*

For an example for the efficiency of this algorithm see Figure 2. For more on this topic refer also to [4].

APPLICATION TO TIME FREQUENCY MASKING

Masking can be defined generally as the situation, where the presence of one stimulus, the masker, decreases the response to another stimulus, the target. In psychoacoustics both frequency [23] and temporal masking [11] is well-known.

A simple model for an extension of masking to the time-frequency domain is presented here. The STFT of the signal is convolved with a 2D kernel function. This kernel function is, in first approximation, just a pyramidal function, which combines a simple triangular function for simultaneous masking (an approximation for the excitation pattern on the basilar membrane as used in [9]) with a just as simple function for the temporal effect. For the general idea see Figure 3. Using the result of this 2D-convolution as threshold for the magnitude of the signal is equivalent to a Gabor multiplier with coefficients in $\{0, 1\}$. The Bark scale is a empirically determined frequency scale better fitted to the human auditory perception than the Hz scale. If the analysis is done on this scale, the Gabor filter has to be irregular.

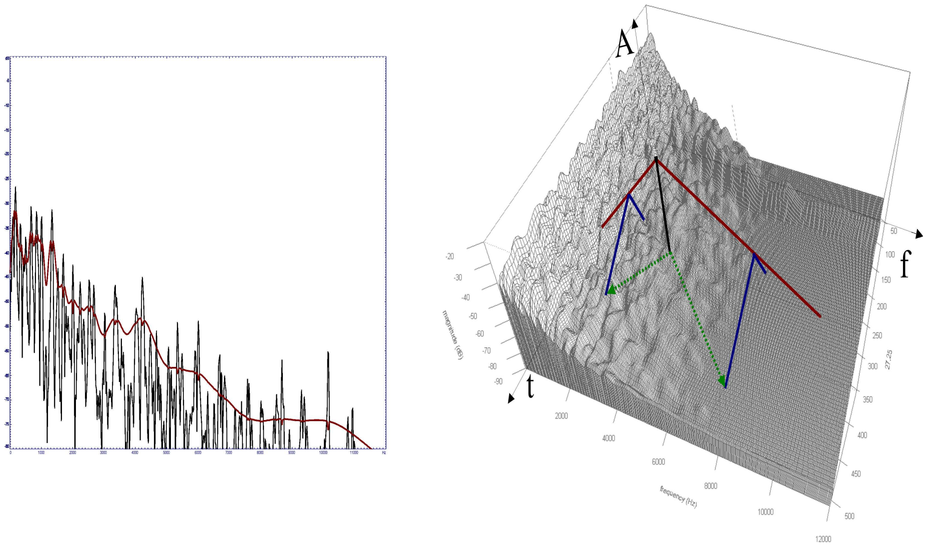

Figure 3: *Time frequency masking idea. (Left:) Single spectrum with threshold function [9] (red). (Right:) STFT. Through the combination of temporal effects (blue) and the spreading function (red) a time frequency masking effect (green) of one point (black) in the time-frequency plane is modeled. The whole STFT is then convolved with this 'pyramidal' function.*

CONCLUSION

Many of these topics are still work in progress. Many results have been shown in [2] and representative ones have been mentioned here. But others are still to be investigated. For example, for general frame multipliers the connection to the notion of weighted frames [5] is currently investigated. Or, more applied, in cooperation with the CNRS Marseille psychoacoustical experiments are currently underway to investigate the time frequency masking effect of a single Gabor atom of Gaussian shape.

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