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ROBUST RESPONSES' CONVERGENCE FOR TIME INTEGRATION OF NONLINEAR EQUATIONS OF MOTION IN LENGTHY TIME INTERVALS

Aram Soroushian

Structural Engineering Research Center,
International Institute of Earthquake Engineering and Seismology (IIEES), No. 26, Arghavan, North Dibaji, Farmanieh, Tehran, Iran, a.soroushian@iiees.ac.ir, Aram.Soroushian@gmail.com

Abstract

Time integration is the most versatile approach to analyze the dynamic behaviour of structural systems. Nevertheless, the resulting responses are approximations and need to converge to exact responses. In presence of nonlinearity, the responses do not exhibit reliable convergence. Many studies to overcome this phenomenon are reported in the literature. A question still debatable is the efficacy of the proposed methodologies when dynamic systems are time integrated in lengthy time intervals. The objective of this paper is to demonstrate that a methodology recently proposed by the author of this paper can reliably maintain convergence even for nonlinear dynamic systems studied in lengthy time intervals.

INTRODUCTION

Due to the complicatedness of structural systems, the behaviours of many systems need to be considered as nonlinear. Furthermore, in regions with high seismic risk, considering inertial forces is of high importance. (Specifically, though yet not explicitly stated in the design codes; nonlinear dynamic analysis should be taken into account, when studying the actual seismic behaviour of buildings structural systems; see [43].) A broadly accepted approach to analyze structural systems with nonlinear dynamic behaviour is to discretize the models defining the dynamic equilibriums in space [1,6,21], and then, to time integrate the resulting mathematical models below,

$$\begin{array}{ll} \mathbf{M}\ddot{\mathbf{u}}(t)+\mathbf{f}_{\text{int}}=\mathbf{f}(t) & 0\le t<T\,, \\ \text{Initial Condition}: & \mathbf{u}(0)=\mathbf{u}_0\,,\quad \dot{\mathbf{u}}(0)=\dot{\mathbf{u}}_0\,,\quad \mathbf{f}_{\text{int}}(0)=\mathbf{f}_{\text{int}_0} \\ \text{Additional Constraints}: & \mathbf{Q}\,. \end{array} \qquad (1)$$

In Eq. (1), $\mathbf{M}$ stands for the mass matrix; $\mathbf{f}_{\text{int}}$ and $\mathbf{f}\,(t)$ respectively express the vectors of internal forces and external excitations; $\mathbf{u}(t)$, $\dot{\mathbf{u}}(t)$, and $\ddot{\mathbf{u}}(t)$ denote the vectors of displacement, velocity, and acceleration, respectively; $\mathbf{u}_0$, $\dot{\mathbf{u}}_0$, and $\mathbf{f}_{\text{int}_0}$ imply the initial status of the mathematical model; and $\mathbf{Q}$ represents some restricting conditions, e.g., problems involved in impact or elastic-plastic behaviour [23,49]; all with respect to the degrees of freedom set for the system under consideration.

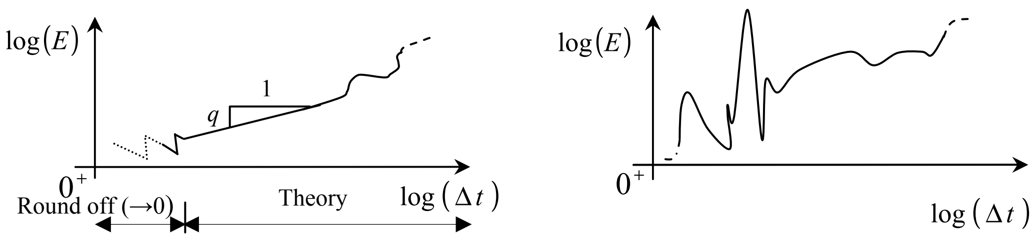
Concentrating on the time integration stage noted above; for the general case of multi-degree of freedom systems involved in nonlinearity [14,19], the formulation can not be exact. Approximate analyses necessitate thorough attention to the concept of convergence [20]. To study the convergence of the responses computed by a time integration method, it is conventionally accepted to consider

$$\lim_{\Delta t \to 0} U_a = U_e \qquad \Delta t > 0 \ , \tag{2}$$

or, in view of the definition of error [33], i.e., $E = \|U_a - U\|$, study the validity of

$$\lim_{\Delta t \to 0} E = 0 \, , \tag{3}$$

where, U_a denotes an arbitrary component of the computed response, U stands for the corresponding exact value, and Δt is a parameter controlling the sizes of all time steps in the integration process. For linear problems, the requirements of convergence are set since decades ago [5,48]. Nevertheless, in presence of nonlinearity, convergence shortcomings are reported by many researchers [4,8,9,10,34,39,42,50]. Figures 1.a and 1.b display the difference between responses convergence in absence



a) Linear *b) Nonlinear*

Figure 1- a) Typical changes of error for practical time steps

and presence of nonlinearity, schematically (q in Fig. 1.a denotes the order of accuracy also known as the rate of convergence [26,48]). In spite of the vast number of studies carried out on this subject [3,4,11,12,15,22,24,27,32], the convergence is not still reliably proper (Fig. 1.a), when the integration interval is lengthy [25]. In view of the fact that one of the recent methodologies for overcoming convergence shortcomings is presented in the Ph.D. Thesis of the author [37], the objective of this paper is to demonstrate that the new methodology does not lose its effectiveness when time integration is carried out in lengthy time intervals. To attain this objective, first, the methodology is briefly reviewed. Then, in view of an example from the literature, the methodology is applied to problems studied in lengthy time intervals, and finally, after a short discussion, the paper is closed with a set of brief conclusions.

THE METHODOLOGY

The author of this paper has recently proposed a new methodology for preserving responses proper convergence (Fig. 1.a) in presence of nonlinearity [18,17,37,40,45]. For the sake of the continuity of the discussion, the methodology is briefly explained below.

Considering conventional time integration of Eq. (1), the approximate computation is at all time instants involved in two sources of errors [35,37,42];

- The errors induced by the approximate formulation of the integration methods.
- The residual errors of nonlinearity solutions.

In order to maintain responses convergence in presence of nonlinearity, it is essential to force the effects of the above two sources to change consistently [16,37,41,42]. In addition, computational facilities can not provide complete accuracy [13,37,41]. Therefore, though at each nonlinearity-detected time step, nonlinearity iterative solutions should end with

$$l = 1, 2, 3, \ldots k-1 \qquad \|\delta_l\| > \bar{\delta} \quad ; \qquad \|\delta_k\| \leq \bar{\delta} \;, \tag{4}$$

where, δ_l implies a component (generally one number) of the residual of the nonlinearity solution after l iterations, $\bar{\delta}$ is a positive constant as the nonlinearity tolerance, and $\| \;\|$ stands for an arbitrary norm [31]; and, in general,

$$\delta_k \neq 0 \;, \tag{5}$$

there exists a close vicinity of zero, i.e. B, we can not decrease the nonlinearity residual to that extent [7,46],

$$\exists\, \varepsilon > 0 \quad \delta_k \notin B(0, \varepsilon) \;. \tag{6}$$

This implies that, when at a nonlinearity-detected time step, $\bar{\delta}$ is very small; we may not succeed to arrive at Eq. (4), even after many iteration. The consequence can be both additional round off error by each iteration, and also stopping at that time step (not proceeding along the time axis) [20,31,37]. An accepted approach to overcome this shortcoming is to restrict the number of nonlinearity iterations [37,46], i.e. k, by

$$k \leq K \;. \tag{7}$$

Therefore, in time integration of Eq. (1), Δt, $\overline{\delta}$, and K are algorithmic parameters, and it is essential to cause numerical consistence among the effects of these three parameters [35,37,42,43]. In this regard, first, Eq. (7) should be replaced with [37],

$$l = 1, 2, 3, \ldots k-1 \quad \log_{10}\left|\frac{S_l}{s_l}\right| \leq \bar{k} - 1 \quad ; \quad \log_{10}\left|\frac{S_k}{s_k}\right| > \bar{k} - 1 . \tag{8}$$

In Eq. (8), S_i and s_i ($i = 1,2,3\ldots k$) denote the total value of the response component in correspondence to δ and the corresponding increment in the last iteration, respectively, and both after i iterations; and $\bar{k}$ is a parameter implying the computational facilities, i.e., single or double precision, see [13].

Next, in order to control the two theoretical sources of error talked about in the start of this section and maintain responses proper convergence in presence of nonlinearity, the new methodology records the residuals of nonlinearity solution, i.e. δ_k, all along the time interval; and considers an arbitrary discretization of the total temporal-spatial space, i.e. Ω, [37,40,45]. If we define this discretization by

$$\bigcup_i {}_i\Omega = \Omega \quad , \quad \forall \; j, k \quad {}_j\Omega \cap {}_k\Omega = \varnothing , \tag{9}$$

where, ${}_i\Omega$ is an arbitrary segment of Ω identified by i as the left subscript; and control the sizes of all the time step with Δt, the nonlinearity tolerances that when being applied in segment (subspace) ${}_k\Omega$ for all k, yield proper convergence in time integration with other Δt can be determined from

$${}_k\overline{\delta'} = \left(\underset{{}_k\Omega}{\mathrm{Min}} \left| {}_m\delta \right| \right) \left(\frac{\Delta t'}{\Delta t} \right)^q . \tag{10}$$

In Eq. (10), $\Delta t'$ is a parameter controlling the time step sizes in the second analysis ($\Delta t' \neq \Delta t$). In brief, comparing Fig. 1.a with the schematic error changes in presence of nonlinearity (Fig. 1.b), we can apply Eqs. (8), (9), and (10) to eliminate the undesired effects of nonlinearity on convergence and modify Fig. 1.b to Fig. 1.a.

PERFORMANCE AT LENGTHY TIME INTERVALS

With regard to the explanations in the previous sections and the step-by-step nature of the time integration methods, it seems reasonable that, when the time intervals corresponding to the segments ${}_i\Omega$ in Eq. (9) are all set equal to the T in Eq. (1), the methodology will maintain proper convergence (Fig. 1.a) regardless of *T*. This is demonstrated in the next section.

ILLUSTRATIVE EXAMPLE

The efficacy of the methodology presented in the previous section along ordinary time intervals is demonstrated by several examples [17,36,40,44,45,47]. One of these examples is as stated below:

$$\begin{gathered} M\left(\ddot{u}+\ddot{u}_g(t)\right)+f_{\text{int}}=0\ ,\quad u(t=0)=\dot{u}(t=0)=f_{\text{int}}(t=0)\ ,\quad T:\quad 0\le t<10 \\ M=1\times 10^5\ \ \text{kg},\quad C=0\ \ \text{N s/m},\quad f_{\text{int}}=C\,\dot{u}+f_s \\ f_s=\text{Linear Elastic}-\text{Plastic with Kinematic Hardening} \\ K=1\times 10^5\ \ \text{N/m},\ K_{plastic}=1\times 10^4\ \ \text{N/m},\ u_y=1\times 10^{-2}\ \ \text{m} \\ \ddot{u}_g(t)=\text{N}-\text{S accelerati on component of the El Centro strong motion } [14]. \end{gathered} \tag{11}$$

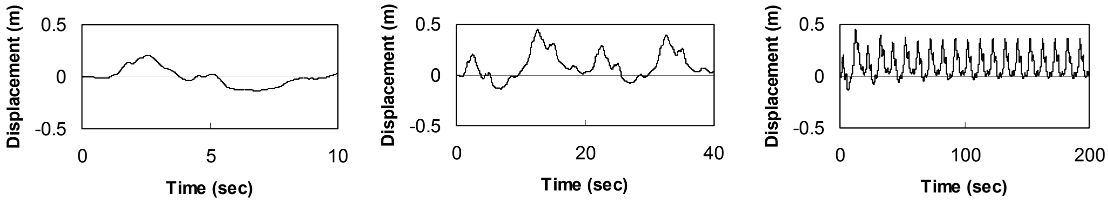
In order to convert the above problem to a new problem with lengthier time interval, first, we should define a parameter to measure the elongation of the time interval. As an straight forward choice, we can use

$$0\le t<T_{new}\ ,\quad T_{new}=a\,T\qquad a>1, \tag{12}$$

where, T and T_{new} denote the length of the time intervals in the original problem, e.g. Eq. (1), and the problem with lengthier time interval, respectively; and a is a parameter defining the lengthiness of the time interval compared to the time interval of the original problem. Considering positive integer values for a, the excitation along the T_{new} time interval can be defined by repeating the original excitation for a times sequentially along the time axis. Applying the above approach and considering $a=4, 20$ results in three problems defined by Eq. (11), (12), and either of

$$1)\quad a=1\qquad\qquad 2)\quad a=4\qquad\qquad 4)\quad a=20\,, \tag{13}$$

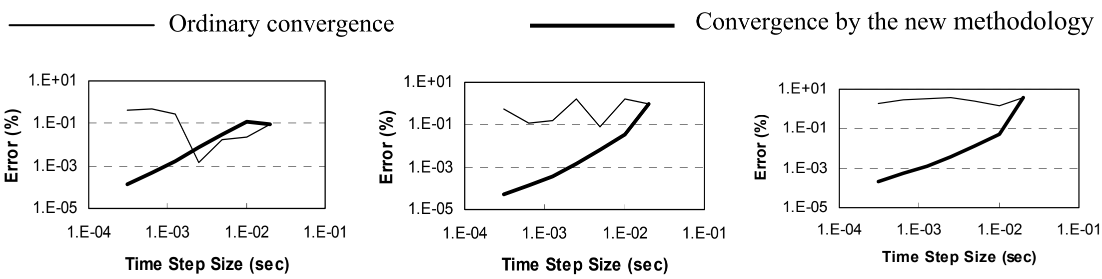
from which the last two can be considered as problems with time intervals lengthier than the time interval of the original problem. (Figure 2 illustrates the almost exact response of these three problems.) To study the efficiency of the new methodology, the above mentioned three problems are analyzed with the Newmark's average acceleration method [30] (average acceleration is the integration method suggested in



a) a = 1 *b) a* = 4 *c) a* = 20

Figure 2 – The almost exact response of the problems defined by Eqs. (11), (12), and (13)

the literature for problems involved in nonlinearity [2]), double precision accuracy [13], and the Fractional-Time-Stepping nonlinearity solution method [28,29,38] with a relative displacement nonlinearity tolerance equal to 0.001. The analyses are then repeated several times, each with halving the time steps. Nonlinearity iterations are controlled by once Eqs. (8) and (10), while ${}_{i}\Omega$ is set equal to the temporal-spatial space consisted of the total time interval and the single degree of freedom; and then once, by Eqs. (4) and (7), while $\bar{\delta}$ (0.001 for relative displacement) and $\bar{k}$ ($\bar{k} = 5$) are constant for all analyses. Considering the errors of the displacements at the end of the time intervals as the criterion for studying convergence, Figs. 3 display the corresponding convergence plots. Figures 1 and 3 apparently reveal that the methodology explained in this paper can reliably maintain the proper convergence trend (Fig. 1.a), even for nonlinear problems studied in lengthy time intervals.



a) a = 1 *b) a* = 4 *c) a* = 20

Figure 3 – Convergence trends for final displacements, in time integration of Eqs. (11), (12), and (13)

CONCLUSIONS

By controlling the iterations of nonlinearity solutions according to a recent methodology [31-34,35,41,42,47], responses generated by time integration can converge properly, even in presence of nonlinearity in lengthy time intervals. Further research on the subject, especially for other types of nonlinearities, seems instructive.

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