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A STOCHASTIC 3D-MODEL FOR SIMULATING VIBRATIONS IN SOIL LAYERS

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Abstract
A major problem for the simulation of vibrations in soil is that material parameters cannot be measured exactly. To avoid this problem stochastic models can be used, but these have the general drawback that the linear system of equations which has to be solved is very large and thus a lot of computer and memory resources are needed. We present ideas for a model that allows the use of stochastic parameters, but also reduces the high costs of solving the system of equations with a special iteration scheme.

INTRODUCTION

Strategies that deal with annoying vibrations caused, for example, by railway lines or heavy machinery in industrial plants become more and more important, especially when buildings are to be constructed near such vibration sources. Numerical models that are able to predict the level of vibrations in the ground can reduce costs for planning and construction of such buildings. Up to now such models are based on deterministic methods or perturbation theory (see for example [1, 2, 3]) where only small perturbations of the material parameters are allowed. But in soil dynamics parameters which describe the soil properties cannot be measured exactly and the results may vary to a large extent. In order to quantify this variability stochastic models, i.e. models that use stochastic material parameters, should be used, but in general such models have one major drawback: The matrix involved is very large, and therefore an efficient solution of the problem with standard numerical tools is not possible. In this paper we introduce some ideas for a stochastic 3D-model which uses a special iteration scheme that allows us to decouple the large scale system into smaller subsystems, thus making an efficient numerical solution possible.

GENERAL SETUP

We assume that it is possible to divide the ground into different horizontal layers. Material parameters for these layers can either be deterministic or stochastic. As an additional feature it is possible to add a fluid layer on top and - to prevent reflections at the bottom- a halfspace layer at the bottom.

All layers except the stochastic ones are modelled using Helmholtz potentials, whereas for the stochastic layers a formulation using finite elements is used. In this paper we will concentrate on the stochastic layers. Ideas for the deterministic case can be found, for example, in [4, 5]. On top of the first layer a load which moves with a speed v along the x-axis is applied. This load oscillates with a frequency f_0 thus simulating some vibrating source. It can either be a point or a rectangular shaped load with length b_1 and height b_2 pointing downwards in the z-direction.

To describe the relation between deformation and external force we use a weak variational formulation:

$$\delta \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} L - 2 f_{\mathrm{Ext}} w_{|z=0} dy dx dt = 0 \tag{1}$$

with

$$L = \int_{-\infty}^{\infty} \frac{2(1-\nu)}{1-2\nu} G(u_x^2 + v_y^2 + w_z^2) + \frac{4\nu}{1-2\nu} G(u_x v_y + v_y w_z + u_x w_z) + \\ + G((u_y + v_x)^2 + (v_z + w_y)^2 + (w_x + u_z)^2) - \rho(u_t^2 + v_t^2 + w_t^2) dz. \tag{2}$$

Here δ indicates the variation, $\boldsymbol{u} = (u, v, w)^T$ is the vector containing the deformations in x, y and z direction. The properties of the soil layers are described with the shear modulus $G(x, y, z, \theta)$, the Poisson ratio ν and their density ρ. It was assumed that only the shear modulus is dependent on the stochastic variable θ. The load is represented by the function f_{Ext}. For simplicity of notation we define $u_x := \frac{\partial u}{\partial x}$, all other derivatives are defined analogically.

STOCHASTICS

To deal with the stochastics of the shear modulus and the deformations we will use the following two decompositions:

Karhunen Loeve Expansion (KLE)

We assume the shear modulus $G(x, y, z, \theta)$ to be a second order Gaussian random process with mean $G_0(z)$ and bounded variance. Therefore it is possible to use the KLE (see [7] for details) to expand G into

$$G(\boldsymbol{x}, z, \theta) = G_0(z) + G_s \sum_i \sqrt{\lambda_i} f_i(\boldsymbol{x}) \xi_i(\theta). \tag{3}$$

For our model we assume that the ξ_i form a series of orthogonal normally distributed random functions, additionally we impose the constraint that the mean of G is only dependent on z. For better readability we define the vector $\boldsymbol{x} := (x, y)$. The $(\lambda_i, f_i(\boldsymbol{x}))$ are eigenvalues of the Fredholm integral equation of the 2$^{\text{nd}}$ kind with the covariance function $C(\boldsymbol{x}_1, \boldsymbol{x}_2)$ as kernel.

To simplify calculations we assume that C can be split in the form $C(\boldsymbol{x}_1, \boldsymbol{x}_2) := C_1(x_1, x_2)C_2(y_1, y_2)$, so $(\lambda_i, f_i(\boldsymbol{x}))$ can be calculated as the product of solutions of the simpler one-dimensional integral equations with C_1 and C_2 as kernels. Ways to calculate these eigenpairs numerically can be found, for example, in [6, 7, 8].

In the next sections we will often use the terms *deterministic* and *stochastic* parts of the equation. With Eq. (3) it is possible to split G into a deterministic part containing only the mean $G_0(z)$ and a stochastic part containing the rest. Similarly to this the whole system can be split up into two such parts.

Chaos polynom expansion

The randomness in the material parameter also causes a randomness in the deformations. To handle this we use an expansion into a series of Hermite Polynomials $\Gamma_i(\boldsymbol{\xi})$ (see also [7, 9]):

$$\boldsymbol{u}(x, y, z, t, \theta) = \sum_i u^{[i]}(x, y, z, t)\Gamma_i(\boldsymbol{\xi}). \tag{4}$$

The vector $\boldsymbol{\xi}$ is build up by the different ξ_i from Eq. (3).

The Hermite polynomials are orthogonal with respect to the L_2-product with the measure $e^{-\frac{\boldsymbol{\xi}^2}{2}}$, i.e.

$$\mathbb{E}(\Gamma_i \Gamma_j) = \int_{-\Omega} \Gamma_i(\boldsymbol{\xi})\Gamma_j(\boldsymbol{\xi}) e^{-\frac{\boldsymbol{\xi}^2}{2}} \mathrm{d}\boldsymbol{\xi} = 0, \ \forall i \neq j. \tag{5}$$

THE LINEAR SYSTEM

The remaining functions $\boldsymbol{u}^{[i]}(x,y,z,t)$ from Eq. (4) are expanded using 2^{nd} order B-splines $h_j(z)$:

$$\boldsymbol{u}^{[i]}(x,y,z,t) = \sum_j \boldsymbol{u}_j^{[i]}(x,y,t)h_j(z). \tag{6}$$

After a Fourier transformation $(x,y,t) \to (k,\ell,\omega)$, an approximation of the integrals (now with respect to k, ℓ and ω) with sums and variation with respect to the unknowns $\hat{\boldsymbol{u}}_j^{[i]}(k_{i_1},\ell_{i_2},\omega_{i_3})$ we can derive a linear system of equations

$$\hat{\boldsymbol{K}}\hat{\boldsymbol{u}} = \hat{\boldsymbol{f}}_{\text{Ext}} \tag{7}$$

with stiffness matrix $\hat{\boldsymbol{K}}$, a vector of the unknown deformations $\hat{\boldsymbol{u}}$ and a vector $\hat{\boldsymbol{f}}_{\text{Ext}}$ for the external load. As we shall see later on, k_{i_1} and ℓ_{i_2} can be chosen arbitrarily, but there is a strong link between k_{i_1} and ω_{i_3}.

Even for a small problem with $(k_{i_1},\ell_{i_2}) \in [-6,6]\times[-6,6]$, 10 layers, an expansion length of 4 for the KLE and a maximum order of 2 for the Hermite-Polynomials (which means we have to consider 15 different Hermite polynomials), the dimension of the system would be $n = 76050$ for one single frequency. Assuming $\mathcal{O}(n^3)$ for a LU decomposition, we would need approximately $4 \cdot 10^{14}$ operations just for the LU decomposition of $\hat{\boldsymbol{K}}$. Unfortunately as we will see later, $\hat{\boldsymbol{K}}$ is not sparse, so solvers designed for such matrices cannot be used either.

Iteration scheme

To solve Eq. (7) efficiently an iterative scheme is used. The main idea behind this scheme is to split the system into a matrix $\hat{\boldsymbol{K}}_{\boldsymbol{d}}$ containing all deterministic parts and a stochastic matrix $\hat{\boldsymbol{K}}_{\boldsymbol{s}}$. The solution of the whole system is now calculated using

$$\begin{aligned} \hat{\boldsymbol{K}}_{\boldsymbol{d}}\hat{\boldsymbol{u}}_0 &= \hat{\boldsymbol{f}}_{\text{Ext}} \\ \hat{\boldsymbol{K}}_{\boldsymbol{d}}\hat{\boldsymbol{u}}_n &= -\hat{\boldsymbol{K}}_{\boldsymbol{s}}\hat{\boldsymbol{u}}_{n-1} + \hat{\boldsymbol{f}}_{\text{Ext}}. \end{aligned} \tag{8}$$

This scheme has the advantage that we can exploit the special block-diagonal structure of $\hat{\boldsymbol{K}}_{\boldsymbol{d}}$ to simplify calculations: Instead of one large system we have to solve several smaller subsystems (one “small” block for each wavenumber, each Hermite polynomial and each frequency). This special structure can be explained by the way the mean of G was split into deterministic and stochastic part.

When making a Fourier transformation with respect to $\boldsymbol{x} = (x,y)$ only products of two functions dependent on $\boldsymbol{x}$ have to be transformed (see Equ. (2) because we assumed that G_0, ν and ρ are independent of $\boldsymbol{x}$. Since a weak formulation is used we can apply Plancherel’s theorem:

$$\int_{\mathbb{R}^2} G_0 a(\boldsymbol{x})b(\boldsymbol{x})d\boldsymbol{x} = \int_{\mathbb{R}^2} G_0\hat{a}(\boldsymbol{k})\hat{b}(-\boldsymbol{k})d\boldsymbol{k}\ . \tag{9}$$

Here a and b are some arbitrary functions, $\hat{a}$ and $\hat{b}$ are their Fourier transforms.

In the stochastic part we have the additional functions f_i from the KLE (see Eq. (3)). Thus Plancherel's theorem can only be used in combination with a convolution

$$\int_{\mathbb{R}^2} f_i(\boldsymbol{x})a(\boldsymbol{x})b(\boldsymbol{x})d\boldsymbol{x} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \hat{f}_i(\boldsymbol{k}-\boldsymbol{k}')\hat{a}(\boldsymbol{k'})\hat{b}(-\boldsymbol{k})d\boldsymbol{k}'d\boldsymbol{k} \tag{10}$$

Compared to Eq. (9) where only functions are coupled at the same wavenumber $\boldsymbol{k}$, we have an additional integral in the stochastic case (Eq. 10), that implies a full coupling for all wavenumbers $\boldsymbol{k}$ and $\boldsymbol{k}'$.

An similar argument can be used when calculating the expectation value. In the deterministic case we can use the orthogonality of the Hermite polynomials

$$\mathbb{E}(ab) = \mathbb{E}\left(\sum_i a^{[i]}\Gamma_i \sum_j b^{[j]}\Gamma_j\right) = \sum_i\sum_j a^{[i]}b^{[j]}\mathbb{E}(\Gamma_i\Gamma_j) = \sum_i a^{[i]}b^{[i]}\mathbb{E}(\Gamma_i^2). \tag{11}$$

In the stochastic case there is the additional random function $\boldsymbol{\xi}$ and the mixed expectation values $\mathbb{E}(\boldsymbol{\xi}\Gamma_i\Gamma_j)$ are in general non-zero. So compared to the deterministic case there is a strong coupling between coefficients for all Hermite polynomials.

MOVING LOADS

As already mentioned an external load that moves with a velocity v along the x-axis and that vibrates with a frequency f_0 can be used with the model. It can either be a pointload or a rectangular load with length b_1 and width b_2. In the first case the load is described as

$$f_{\text{Ext}} = P\delta(x-vt)\delta(y)\cos(2\pi f_0 t)\ , \tag{12}$$

the second load is defined with

$$f_{\text{Ext}} = \frac{P}{b_1 b_2}\Pi\left(\frac{x-vt}{b_1}\right)\Pi\left(\frac{y}{b_2}\right)\cos(2\pi f_0 t). \tag{13}$$

Here $\delta(x)$ denotes the Dirac Delta function, $\Pi(x)$ is the rectangle function

$$\Pi(x) = \begin{cases} 0 & |x| > \frac{1}{2} \\ \frac{1}{2} & |x| = \frac{1}{2} \\ 1 & |x| < \frac{1}{2} \end{cases} . \tag{14}$$

The calculations of the deformations are done inside the region $(x, y) \in [-a_x, a_x] \times [-a_y, a_y]$, outside this interval the deformations are continued periodically. After a

Fourier transformation we get

$$\hat{f}_{Ext} = \frac{P}{8a_x a_y}(\delta(\omega + \frac{kv}{2a_x} + f_0) + \delta(\omega + \frac{kv}{2a_x} - f_0)) \tag{15}$$

for the pointwise load, for the rectangular load we have

$$\hat{f}_{Ext} = \frac{P}{8a_x a_y}\mathrm{sinc}\left(\frac{b_1 \pi k}{2a_x}\right)\mathrm{sinc}\left(\frac{b_2 \pi \ell}{2a_y}\right)(\delta(\omega + \frac{kv}{2a_x} + f_0) + \delta(\omega + \frac{kv}{2a_x} - f_0)) \ . \tag{16}$$

Combining the above equations with Eq. (2) we can see that the important part of $\boldsymbol{f}_{\mathrm{Ext}}$ consist of the factors

$$\hat{w}_j^{[i]}(-k_{i_1}, -\ell_{i_2}, \frac{k_{i_1} v}{2a_x} - f_0) + \hat{w}_j^{[i]}(-k_{i_1}, -\ell_{i_2}, \frac{k_{i_1} v}{2a_x} + f_0) \ . \tag{17}$$

This means that for each wavenumber k_{i_1} there exist only two frequencies for which the right hand side of the small subsystems are nonzero, which again reduces the computational effort. In Eq. (17) we can also see the already mentioned coupling between the wavenumbers and the frequencies.

EXAMPLE

As an example to test our model we use a setup with 40 deterministic layers, with one fluid layer on top and one halfspace layer at the bottom to prevent reflections. Material parameters for this example are given in Table 1, as external load we use a force moving with velocity $v = 60$m/s and with a frequency $f_0 = 100$Hz. The wavenumber grid consist of 33×33 points, the bounds in the x and y directions are set to $a_x = 30$ and $a_y = 30$. Figure 1 shows the deformations with respect to the

Layer	G_0	G_s	ρ	ν	d
Fluid	2e9+2e6i		1000		1
Stoch.	2e8+2e7i	1e6+1e5i	1800	0.3	0.2×20
Halfsp.	2e8+1e7i		1800	0.3	

Table 1: Material parameters for the example: G_0 and G_s are mean value and variance factor of the shear modulus in N/m^2, ρ the density in kg/m^3, ν the Poisson ratio and d height of the layer in m.

z-direction of the first stochastic layer. The excitations at the far right edge of the grid are due to our periodicity assumption. A similar plot for the standard deviation of the deformation is depicted in Figure 2. Again please note that the excitations at the right edge are due to the periodicity of the system.

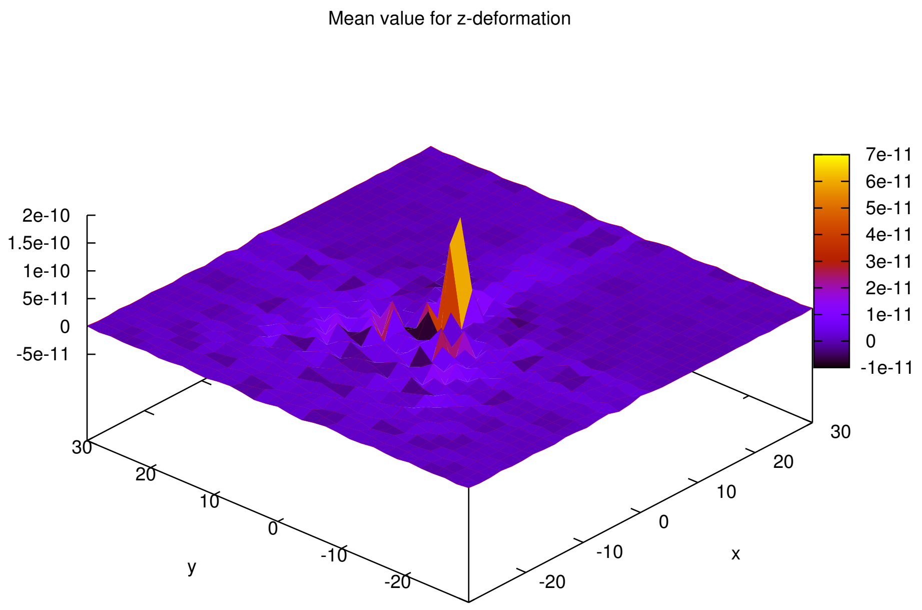

Figure 1: Mean value of the deformation in the z-direction for the first stochastic layer after 21 time steps

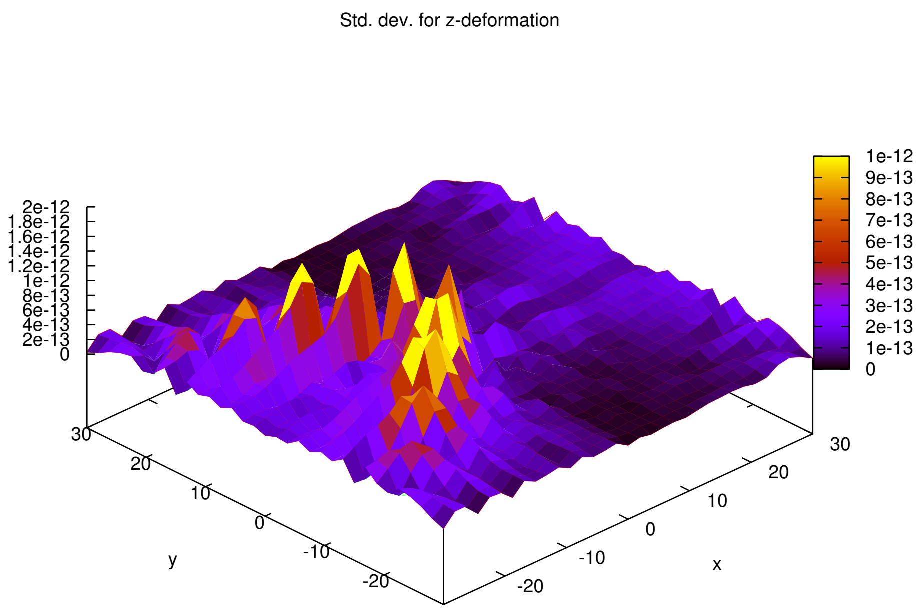

Figure 2: Standard deviation of the deformation in the z-direction for the first stochastic layer after 21 time steps

SUMMARY

A 3D model to calculate vibrations in soil layers was presented. With this model it is possible to use a stochastic shear modulus which normally would increase the computational cost for calculations to a high degree. An iteration scheme was introduced which reduceed this calculation costs. Finally some results for a test example were presented. This project was supported by the Austrian Science Fond, FWF: P-16224-N07.

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