A Fourier Transform 3D-Model for Wave Propagation in layered orthotropic Media

A Fourier Transform 3D-Model for Wave Propagation in layered orthotropic Media [1]

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1 ABSTRACT

In order to assess and prevent noise emmission in the environment of traffic routes, prognosis models for wave propagation are of interest for many applications.Guided by experience in railway traffic simulation, this paper introduces a new model for wave propagation and simulation in layered orthotropic media. The model uses a component-wise Fourier transform to obtain the general solution within one layer; then, balances at layer boundaries are used to obtain a grid of stress / displacement vectors.The accuracy of the model is tested, and results are presented. Some ideas on implementation and acceleration are presented as well.

PACS Code: 4320Bi Mathematical theory of wave propagation (linear acoustics)

2 INTRODUCTION

The simulation of vibrations induced by machinery and traffic has become of increasing concern within the last decades, due to frequent construction of high speed trains and mass rapid transport systems [1]. Theoretical studies on the propagation of waves in anisotropic layered media have been an important field of research for many years, and have been of particular use for applications in geophysics and seismology [2, 3]. Therefore, much research has been done on wave propagation in anisotropic layered media [4], however, layered orthotropic media has rarely been focused upon. In order to derive a suitable model for applications, this paper assumes the soil to consist of orthotropic layers, and an orthotropic halfspace bounding from below. Waubke [5] has also suggested extensions and developments for this model in earlier papers.

This paper draws these ideas together, and offers some concrete numerical results. In the first section, a model of wave propagation is derived by Fourier transform of the classical wave equation. The model is based on a Fourier transform of the wave equations, resulting in a system of polynomial equations. Backward transformation in the vertical direction is used to handle the boundary conditions. The balance of stresses and displacements at the layer boundary delivers a system of linear equations, yielding the stress and displacement level at the layer boundaries as functions $\hat{u}(k_x, ky, z, \omega)$ and $\hat{\sigma}(k_x, k_y, z, \omega)$. Thus, the displacement and stress levels can be calculated at arbitrary points (k_x, k_y, z, ω); backward Fourier transform yields a grid in (x, y, z, ω).This model applied to a system of layers gives a transport matrix for the layer boundaries. It yields a system, established in section 3, whose solution permits the computation of stress and displacement levels at arbitrary points within the media. The advantage of this approach is its numerical stability, which permits the calculation of displacement and stress levels for large distances from the point of excitation. At first, the focus lies on one layer and the equations describing the process are derived. The strain vector ϵ is a vector of local derivatives of the displacement vector u, denoted by $\epsilon := D(\partial_x, \partial_y, \partial_z)u$. The soil is assumed to consist of n orthotropic layers with widths $d_1, \dots d_n$. For each layer, the following parameters are needed: item the shear modulus G_{xy}, G_{yz}, G_{zx}, the Young's modulus E_x, E_y, E_z, the Poisson ratios $\nu_{xy}, \nu_{yz}, \nu_{zx}$, the density ρ and width d. Additionally, homogenous behavior in the horizontal directions is assumed: $E_y = E_x$, $G_{yz} = G_{zx}$, and $\nu_{yz} = \nu_{zx}$ hold. The stress vectors σ can be computed as they correspond with ϵ via the elasticity matrix, $\sigma = E\epsilon$; the matrix E isof block diagonal form with

$$E_{11} := \begin{pmatrix} \frac{1}{E_x} & -\frac{\nu_{xy}}{E_x} & -\frac{\nu_{zx}}{E_z} \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_x} & -\frac{\nu_{zx}}{E_z} \\ -\frac{\nu_{zx}}{E_z} & -\frac{\nu_{zx}}{E_z} & \frac{1}{E_z} \end{pmatrix}^{-1}, \quad E_{22} := \mathrm{diag}(2G_{xy}, 2G_{zx}, 2G_{zx}) \tag{1}$$

The equilibrum of forces leads to the equation

$$\underbrace{\left((\partial_x, \partial_y, \partial_z) \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{yz} & \sigma_{zz} \end{pmatrix} \right)^T}_{=:B\sigma} - \rho\partial_t^2 \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} + \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2}$$

where b is some external force. As Eq. (2) is linear in (u_x, u_y, u_z), arbitrary solutions can be calculated by finding the homogenous and particular components of the solution. We denote the Fourier transform $\hat{f}$ of a function $f(\alpha)$ by $\mathcal{F}_{\alpha,k_\alpha}(f)(k_\alpha)$ The Fourier back transform is denoted by $\mathcal{F}^{-1}_{k_\alpha,\alpha}$.The displacement vector $u := u(x, y, z, t)$ is Fourier transformed in all directions:

$$\hat{u}(k_x, k_y, k_z, \omega) := \mathcal{F}_{x,k_x}\left(\mathcal{F}_{y,k_y}\left(\mathcal{F}_{z,k_z}\left(\mathcal{F}_{t,\omega}\left(u\right)\right)\right)\right)(k_x, k_y, k_z, \omega). \tag{3}$$

Thus, the Differential operator D is substituted by the polynomial operator $\hat{D}$. Simple equations are obtained, expressing the link of the transformed displacement vector and the transformed stress vector: $\hat{\sigma} = E\hat{\epsilon} = E\hat{D}\hat{u}$, where $\hat{D} = \hat{D}(k_x, k_y, k_z)$ is the transformed differential operator D. Note that $E\hat{D}$ is symmetric. The wave equation takes the form

$$\hat{B}E\hat{D}\hat{u} + \rho\omega^2\hat{u} =: \hat{A}\hat{u} = \hat{b}. \tag{4}$$

Nontrivial homogenous solutions exist in the case when $\det(\hat{A})(k_x, k_y, k_z, \omega) = 0$. In order to cope with the boundary conditions, it is useful to implement a grid in (k_x, k_y, ω). Therefore k_x, k_y, ω are treated as constants, and are omitted in some cases, for better readability.

3 DERIVING THE EQUATIONS

3.1 The general form of a solution

Note that the matrix $\hat{A}$ has some symmetry properties, which we will stress in the subsequent calculations. The condition $\det(\hat{A}) = 0$ is necessary for solutions of $\hat{A}\hat{u} = 0$, $\hat{u} \neq 0$ to exist. So, the next task is to compute the values of k_z such that $\hat{A}$ is singular. The values with

$\det(\hat{A}(k_z)) = 0$ are called eigenvalues κ by abuse of notation, the vectors of the corresponding kernels are called eigenvectors, denoted by Ψ differing from the mathematical definition. The polynomial is of degree 6 without odd powers of k_z, and so the substitution $\lambda = k_z^2$ is justified, reducing the degree of the considered polynomial to 3 at the same time reducing calculation effort. Note that the homogeneous solutions u_h and σ_h are nontrivial for $k_z = \kappa_i$ for $i = 1, \ldots, 6$ such that the homogeneous solution u_h and the corresponding stress function σ_h take the form

$$\hat{u}_h(k_z) = \sum_{i=1}^{6} c_{h,i} \Psi_i \delta(k_z - \kappa_i), \quad \hat{\sigma}_h(k_z) = \sum_{i=1}^{6} c_{h,i} E\hat{D}(\kappa_i) \Psi_i \delta(k_z - \kappa_i) \tag{5}$$

for arbitrary coefficients $c_{h,i}$ for $1 \leq i \leq 6$. We define $H(k_x, k_y, \kappa_i, \omega) := E\hat{D}(k_x, k_y, \kappa_i, \omega)$ for better readability, and write $H(\kappa_i) =: H_i$ for the sake of simplicity. Some properties of the eigenvalues can be derived directly from the structure of $\hat{A}$; let $\Psi(k_z)$ be an eigenvector of $\hat{A}(k_z)$. The symmetry of $\hat{A}$ yields

$$A(k_z) = E_3^T A(-k_z) E_3 \qquad \text{for} \quad E_3 = \text{diag}(1, 1, -1). \tag{6}$$

Using this, the eigenvector $\Psi(-k_z)$ can be obtained from $\Psi(k_z)$ by changing the sign of the third component. The effort of the eigenvector calculation is reduced to the half thus. In order to simplify the notation, the eigenvalues and eigenvectors are numbered respectively; so, let κ_i be the i-th eigenvalue of $\hat{A}$, and Ψ_i the corresponding eigenvector, for $1 \leq i \leq 6$.
As the model assumes the media to be layered with respect to the z-direction, it is necessary to evaluate the displacement vectors at specific values for z. Therefore, $\hat{u}$ is transformed back over the z direction,

$$\tilde{u}(k_x, k_y, z, \omega) = \mathcal{F}^{-1}_{k_z, z}(\hat{u}(k_x, k_y, k_z, \omega))(z). \tag{7}$$

Equations for $\tilde{u}$ are derived by performing the transformation. Any homogeneous solution $\tilde{u}_h$ can be expressed as a linear combination of the eigenvectors Ψ_i; For the homogeneous part $\tilde{u}_h$ of the displacement vector, and the homogeneous part of the stress vector $\tilde{\sigma}_h$ this yields with Eq. (5)

$$\tilde{u}_h = \sum_{i=1}^{6} c_{h,i} \Psi_i \exp(j\kappa_i z), \quad \tilde{\sigma}_h = \sum_{i=1}^{6} c_{h,i} H(\kappa_i) \Psi_i \exp(j\kappa_i z), \tag{8}$$

By variation of constants ansatz, the general form of the solution can be expressed as

$$\tilde{u} = \sum_{i=1}^{6} (c_{h,i} + c_{p,i}(z)) \Psi_i \exp(j\kappa_i z), \quad \tilde{\sigma} = \sum_{i=1}^{6} (c_{h,i} + c_{p,i}(z)) H_i \Psi_i \exp(j\kappa_i z) \tag{9}$$

for z-dependent coefficients $c_{p,i}(z)$ from a particular solution.

Figure 1: Treatment of a loaded layer: the layer is split by a virtual layer boundary, at the depth of the impact.

3.2 Calculation of stress and displacement vector at layer boundaries

In order to derive the equations for the particular solution, a discrete impact at at depth d inside a layer is assumed. Concretely, the impact at a point (x_0, y_0, z_0, t_0) with constant direction f is assumed in the form

$$p(x, y, z, t) = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0)\delta(t - t_0)f. \tag{10}$$

The Fourier transform is applied on the $p(z)$. This yields

$$\hat{p}(k_x, k_y, k_z, \omega) = \exp(j(x_0 k_x + y_0 k_y + z_0 k_z + t_0\omega))f. \tag{11}$$

As the model is invariant to translation, the origin of the coordinates (x, y, z, ω) is set to the point of impact, so that $\hat{p} = f$. Note that if the origin is assumed in the middle of the layer, a transformation factor $\exp(z_p)$ is yielded (see Figure 1). A Fourier backward transform gives $\hat{p}$ as right hand side of Eq. (4). It is to be taken into account that the balance of stresses only holds for the components that act at the interface of the layer boundary. Thus, and by Eq. (**??**), only the components $3, 5$ and 6 are important for the stress balance equations. So, the vector $\tilde{\sigma}$ is reduced to dimension 3 by dropping the other components. To be exact, the reduced matrix H_r is defined consisting of the third, fifth and sixth column of H; H is replaced by H_r in Eq. (8).

A layer with width d is considered, loaded with an impact vector $\hat{p}$ in depth d_p. The layer is split at the depth of the impact d_p, and two layers with identical material parameters are derived. The lower virtual layer is treated as an unloaded layer. For the upper virtual layer let $\tilde{u}_u(k_x, k_y, z = -d_p, \omega)$ and $\tilde{u}_m(k_x, k_y, z = 0, \omega)$ be the transformed displacement at the upper and lower boundary, respectively. For the lower virtual layer, let $\tilde{u}_n$ be the displacement vector at the upper boundary and $\tilde{u}_d$ the one at the lower boundary. Evidently, $\tilde{u}_m = \tilde{u}_n$ holds.

$$\tilde{u}_u = \sum_{i=1}^{6} a_i \Psi_i \exp(-j\kappa_i d_p), \qquad \tilde{\sigma}_u = \sum_{i=1}^{6} a_i H_{r,i} \Psi_i \exp(-j\kappa_i d_p) \tag{12}$$

$$\tilde{u}_m = \sum_{i=1}^{6} a_i \Psi_i, \qquad \tilde{\sigma}_m = \sum_{i=1}^{6} a_i H_{r,i} \Psi_i \tag{13}$$

$$\tilde{u}_n = \sum_{i=1}^{6} b_i \Psi_i, \qquad \tilde{\sigma}_n = \sum_{i=1}^{6} b_i H_{r,i} \Psi_i, \tag{14}$$

$$\tilde{u}_d = \sum_{i=1}^{6} b_i \Psi_i \exp(j\kappa_i (d - d_p)), \qquad \tilde{\sigma}_d = \sum_{i=1}^{6} b_i H_{r,i} \Psi_i \exp(j\kappa_i (d - d_p)). \tag{15}$$

The equations for displacements and stress at the point of impact can be established now. As mentioned above, u_n and u_m equate to each other. The stress vectors $\tilde{\sigma}_{r,m}$ and $\tilde{\sigma}_{r,n}$ at the virtual layer boundary differ by $\hat{p}$. The coefficient vector a and b consist of homogenoeous and particular part, $a = a_h + a_p$, $b = b_h + b_p$. The homogeneous parts equate to each other, $a_h = b_h$. This can easily be verified by setting the load vector to zero. The particular part $b_p = 0$ as the lower virtual layer is treated as an unloaded layer. The particular part a_p has to satisfy

$$\tilde{u}_{p,m} = \sum_{i=1}^{6} a_{p,i} \Psi_i = 0, \quad \tilde{\sigma}_{p,m} = \sum_{i=1}^{6} a_{p,i} H_{r,i} \Psi_i = \hat{p}. \tag{16}$$

This gives a_p as the solution of a linear equation

$$M a_p = \hat{p}_0 \qquad \text{with} \quad M := \begin{pmatrix} \Psi_i \\ H_{r,i} \Psi_i \end{pmatrix}_{(i=1,\dots,6)}, \quad \hat{p}_0 := \begin{pmatrix} 0 \\ \hat{p} \end{pmatrix}. \tag{17}$$

The layers are unified again. The coefficient of the displacements and stresses of the unified layers are denoted by $c = c_h + c_p$. The particular component of the solution is given by $c_p = a_p = M^{-1} \hat{p}_0$. For the displacement and stress vectors at the layer boundaries it is to be taken into account that the origin is set to the middle of the layer. This yields a correction factor $\exp(j\kappa_i z_p)$ for the partial components. For an arbitrary z within the layer boundaries, the stress vector $\sigma(z)$ and the displacement vector $\tilde{u}(z)$ is thus computed

$$\begin{aligned} \tilde{u}(z) &= \sum_{i=1}^{6} (c_{h,i} + c_{p,i} \exp(j\kappa_i z_p)) \Psi_i \exp(j\kappa_i (z - \frac{d}{2})), & (18) \\ \tilde{\sigma}(z) &= \sum_{i=1}^{6} (c_{h,i} + c_{p,i} \exp(j\kappa_i z_p)) H_{r,i} \Psi_i \exp(j\kappa_i (z - \frac{d}{2})). & (19) \end{aligned}$$

The factors $\exp(-j\kappa_i (z - d/2))$ are derived from the backward Fourier transform of the i-th component for the upper layer boundary (compare with Eq. (8)); as a consequence, $\exp(\pm\frac{1}{2} j\kappa_i d)$ arise componentwise for the layer boundaries, and $\exp(j\kappa_i z_p)$ for the particular solution. The matrices Θ and Ξ, as well as the vector $\tilde{p}$, are defined as follows, in order to express the displacements and stresses at the layer boundaries:

$$\begin{aligned} \Theta &:= \begin{pmatrix} \Theta_u \\ \Theta_d \end{pmatrix} := \begin{pmatrix} \Psi_i \exp(-\frac{1}{2} j\kappa_i d) \\ \Psi_i \exp(\frac{1}{2} j\kappa_i d) \end{pmatrix}_{(i=1,\dots,6)} & (20) \\ \Xi_r &:= \begin{pmatrix} \Xi_{r,u} \\ \Xi_{r,d} \end{pmatrix} = \begin{pmatrix} H_{r,i} \Psi_i \exp(-\frac{1}{2} j\kappa_i d) \\ H_{r,i} \Psi_i \exp(-\frac{1}{2} j\kappa_i d) \end{pmatrix}_{(i=1,\dots,6)} & (21) \\ \tilde{p} &:= (c_{p,i} \exp(j\kappa_i z_p))_{(i=1,\dots,6)} & (22) \end{aligned}$$

This yields

$$\tilde{u} =: \begin{pmatrix} \tilde{u}_u \\ \tilde{u}_d \end{pmatrix} = \Theta c_h + \begin{pmatrix} \Theta_u \tilde{p} \\ 0 \end{pmatrix} = \tilde{u}_h + \tilde{u}_p, \tag{23}$$

$$\tilde{\sigma}_r =: \begin{pmatrix} \tilde{\sigma}_{r,u} \\ \tilde{\sigma}_{r,d} \end{pmatrix} = \Xi_r c_h + \begin{pmatrix} \Xi_{r,u} \tilde{p} \\ 0 \end{pmatrix} = \tilde{\sigma}_{r,h} + \tilde{\sigma}_{r,p}. \tag{24}$$

Substituting $c_h = \Theta^{-1}(\tilde{u} - \tilde{u}_p)$ into the stress equations, and defining $K := \Xi_r \Theta^{-1}$, it holds that

$$\tilde{\sigma}_r = \Xi_r \Theta^{-1}(\tilde{u} - \tilde{u}_p) + \tilde{\sigma}_{r,p} = K\tilde{u} - K\tilde{u}_p + \tilde{\sigma}_{r,p}, \tag{25}$$

for a loaded layer.The case for an unloaded layer is even simpler, and can easily be deduced from the general case by setting the partial component to zero. Let $\tilde{u}_u$ and $\tilde{u}_d$ be the displacements at the top and bottom boundary of an unloaded layer, respectively, and $\tilde{\sigma}_u$ and $\tilde{\sigma}_d$ the according stress vectors, and K the matrix as above, calculated from the layer data. By setting the partial components to zero in Eq. (25), the equation

$$\begin{pmatrix} \tilde{\sigma}_{r,h,u} \\ \tilde{\sigma}_{r,h,d} \end{pmatrix} = K \begin{pmatrix} \tilde{u}_{u,h} \\ \tilde{u}_{d,h} \end{pmatrix} \tag{26}$$

is derived. As the layer is unloaded, the displacement and stress vectors only consist of their homogeneous part, and the index h can be omitted.

4 BALANCE AT LAYER BOUNDARIES

Now, a system of n layers is considered. For the i-th layer , let the matrices c_i, Θ_i, Ξ_i, H_i, K_i denote the matrices defined in the section before, evaluated for the i-th layer. The layers are numbered in the canonical way, such that $\tilde{u}_1$ is the displacement at the upper boundary of the first layer, and $\tilde{u}_{n+1}$ is the transformed displacement vector at the lower boundary of the lowest layer. The stress vectors are sometimes denoted alternatively as $\tilde{\sigma}_{i,u} = \tilde{\sigma}_i$, if it provides better readability that way. The index r is dropped in σ_r, Ξ_r etc., as the full stress vector will not be considered in this section.
For unloaded layers, the balance of stresses is established by equating the stress term on the lower boundary of layer i with the stress term for the upper boundary of layer $i+1$ (note that $\tilde{u}_{d,i} = \tilde{u}_{i+1} = \tilde{u}_{u,i+1}$). This yields

$$K_{i,d} \begin{pmatrix} \tilde{u}_i \\ \tilde{u}_{i+1} \end{pmatrix} = K_{i+1,u} \begin{pmatrix} \tilde{u}_{i+1} \\ \tilde{u}_{i+2} \end{pmatrix} \tag{27}$$

4.1 Boundaries of loaded layers

The layer k is assumed to be loaded. Let the indices h and p denote the homogeneous and the particular part of the solution. By Eq.s (**??**) and (24), both the homogeneous and the particular component of the stress vector is computed as

$$\tilde{\sigma}_{p,k} = \begin{pmatrix} \Xi_{k,u} \tilde{p}_k \\ 0 \end{pmatrix}, \tag{28}$$

$$\tilde{\sigma}_{h,k} = \Xi_k c_k = \Xi_k \Theta_k^{-1} \tilde{u}_{h,k} = K_k \tilde{u}_{h,k} = \tag{29}$$

$$=: \begin{pmatrix} K_{k,u} \\ K_{k,d} \end{pmatrix} \tilde{u}_{h,k} =: \begin{pmatrix} K_{k,uu} & K_{k,ud} \\ K_{k,du} & K_{k,dd} \end{pmatrix} \tilde{u}_{h,k}. \tag{30}$$

The stress vectors at the layer boundaries equate in the same way, up to the partial component of the solution. With $\tilde{\sigma}_{k,u} = \tilde{\sigma}_{k-1,d}$ it holds that

$$\tilde{\sigma}_{k-1,d} = \tilde{\sigma}_{k,u} = \tilde{\sigma}_{h,k,u} + \tilde{\sigma}_{p,k,u}, \tag{31}$$

and as the stress vector at the upper boundary takes the form $\tilde{\sigma}_{k,u} = \Xi_{k,u}(c_k + \tilde{p}_k)$ (see Eq. (29)), the balance of stresses at the upper loaded boundary takes the form

$$\Xi_{k-1,d} c_{k-1} = \Xi_{k,u}(c_k + \tilde{p}_k) \tag{32}$$

and leads to

$$K_{k-1,d} \begin{pmatrix} \tilde{u}_{k-1} \\ \tilde{u}_k \end{pmatrix} = K_{k,u} \begin{pmatrix} \tilde{u}_k \\ \tilde{u}_{k+1} \end{pmatrix} - K_{k,uu} \tilde{u}_{p,k} + \tilde{\sigma}_{p,k,u}. \tag{33}$$

after substitution $c_i = \Theta_i^{-1}(\tilde{u}_i - \tilde{u}_{p,i})$ for $i = k-1, k$. The vector $\tilde{u}_{p,k}$ is equivalent to $\tilde{u}_p$ from Eq.(23), and $\tilde{\sigma}_{p,k,u}$ is equivalent to $\tilde{\sigma}_{r,p}$ from Eq. (24), for the k-th layer. For the lower boundary, since $\tilde{\sigma}_{k,d} = \tilde{\sigma}_{k+1,u}$, this yields

$$K_{k,d} \begin{pmatrix} \tilde{u}_k \\ \tilde{u}_{k+1} \end{pmatrix} + K_{k,du} \tilde{u}_p = K_{k+1,u} \begin{pmatrix} \tilde{u}_{k+1} \\ \tilde{u}_{k+2} \end{pmatrix}. \tag{34}$$

4.2 Top boundary and bottom boundary

For the displacement $\tilde{u}_1$, which occurs at the boundary between air and soil, the boundary conditions are assumed to be of von Neumann type. The bottom boundary is assumed to be a half space, so we need some considerations to handle its semi-infinity. The discussion is restricted to the unloaded case, as the half space can always be assumed unloaded - be it by introducing a virtual layer boundary. Physical principles for the wave front are applied,and Sommerfeld conditions are included. By assuming that for a monofrequent wave with angular frequency Ω, the wave front $\kappa_i z + \Omega t = 0$ should always travel downward with time, this gives

$$\frac{z}{t} = \mathrm{Re}\left(\frac{z}{t}\right) = -\mathrm{Re}\left(\frac{\Omega}{\kappa_i}\right) > 0 \Leftrightarrow \mathrm{Re}(\Omega\kappa_i) \leq 0. \tag{35}$$

This is interpreted as a condition for the eigenvalues κ_i; the coefficient c_i must be zero if κ_i does not fulfill the condition above. As $\kappa_i = (\pm\sqrt{\lambda}_i)$ for $i = 1, 2, 3$, this eliminates three eigenvalues. As a half space has no lower boundary, the matrices Θ_n, Ξ_n and K_n shrink to size 3×3. At the bottom layer, this yields

$$K_{n-1,d} \begin{pmatrix} \tilde{u}_{n-1} \\ \tilde{u}_n \end{pmatrix} = K_n \tilde{u}_n. \tag{36}$$

4.3 Synthesis

Again, a system of n layers is considered, including the k-th one, which is assumed to be loaded. Putting together the results, the equation to be solved is established defining the supervector $U := (\tilde{u}_1, \ldots, \tilde{u}_n)^T$ (consisting of all displacement vectors). So, it delivers a linear equation $LU = V$, where L is an $n \times n$ block band matrix with 3×3 blocks, and V has 6 nonzero components.

4.4 Obtaining the grids

With the solution U it is possible to compute the stress vector as well as the displacement vector at arbitrary points. Extending the definition of Θ, Ξ and $\hat{p}$ yields

$$\Theta_s(z) := \left(\Psi_{s,i} \exp(j\kappa_{s,i}(z - \frac{d_s}{2}))\right)_{(i=1,\ldots,6)}, \tag{37}$$

$$\Xi_s(z) := \left(H_{s,i}\Psi_{s,i} \exp(j\kappa_{s,i}(z - \frac{d_s}{2}))\right)_{(i=1,\ldots,6)}, \tag{38}$$

(39)

$$\hat{p}_s(z) := \begin{cases} (c_{p,i} \exp(j\kappa_i(d_p - z)))_{(i=1,\ldots,6)} & \text{if} \quad z \le d_p \\ 0 & z > d_p \end{cases}. \tag{40}$$

for the layer with index s and for $0 \le z \le d_s$. The formulas for the displacement vector $u(k_x, k_y, z, \omega)$ and the stress vector $\sigma(k_x, k_y, z, \omega)$ take the form

$$u_s(k_x, k_y, z) = \Theta_s(z)(c_{h,s} + \tilde{p}_s(z)), \quad \sigma_s(k_x, k_y, z) = \Xi_s(z)(c_{h,s} + \tilde{p}_s(z)) \tag{41}$$

as mentioned in Eq. (19). The coefficients $c_{h,s}$ and $\hat{c}_s$ are $c_{h,s} = \Theta_s^{-1}\hat{u}_h$ from Eq. (23). Of course, $\tilde{p}_s(z) = 0$ for unloaded layers.

By implementing a grid in the z-direction, the four dimensional grid $\tilde{u}(k_{x,i_1}, k_{x,i_2}, z_{i_3}, \omega_{i_4})$ can easily be derived by evaluating the values of z. A Fourier back transform in k_x and k_y finally yields a grid $u(x_{i_1}, y_{i_2}, z_{i_3}, \omega_{i_4})$ respectively $\sigma(x_{i_1}, y_{i_2}, z_{i_3}, \omega_{i_4})$, $1 \le i_1, i_2, i_3, i_4 \le n$, which was the intended result.

5 COMMENTS ON TESTING AND IMPLEMENTATION

5.1 Acceleration technique

In order to accelerate the computation, the appearing coefficients are computed in dependency of G_{xy}, G_{zx}, ν_{xy},ν_{zx}, E_x, E_z, and d of the various layers, as they do not depend on the considered grid point. For the determinant of $\hat{A}$, 20 coefficients of double precision have to be computed and saved, and 3 more coefficients for the matrix $\hat{A}$ itself.The symmetries stated in observation are used to accelerate the computation as follows. Let $p_0 := (k_x, k_y, k_z, \omega)$ be an arbitrary point, $p_1 := (-k_x, k_y, k_z, \omega)$ and $p_2 := (k_x, -k_y, k_z, \omega)$. With

$$X(p_1) = E_x(X)X(p_0), \quad X(p_2) = E_y(X)X(p_0), \tag{42}$$

Figure 2:
Left: The relative error of displacements for a four layer system, computed on various points of the $(k_x, k_y) \in [-10, 10] \times [-10, 10]$ plane for $\omega = 10000.0$;
Right: Difference between a four layer system and the same system split within the first layer. (calculated for $(\omega = 10000.0, k_x, k_y \in [-10, 10] \times [-10, 10])$)

appropriate transforming matrices $E_x(X)$, $E_y(X)$ can be found for $X = \kappa$, Ψ,Ξ_r,Ξ. Transforming matrices for M, L, U can be found provided the load vector is a standard normal vector.

5.2 Testing the model

The model is tested in two ways; first, the correctness of the calculations is checked by running a symbolic computation software with the data, and comparing the results to the ones of the C++ program.

Numerical Accuracy For a system of random layers with realistic parameters, accuracy up to the fifth decimal position is detected. See Figure 2 for the results. The computations were done using MAPLE 9.5.

Physical consistency The second test is checking physical consistency. For a system of layers, the solution is computed. By introducing a virtual layer boundary at a random depth, one layer is split in two with identical material parameters. The solution of the new system is computed. The results must coincide at the layer boundaries (if the the virtual layer boundary in the second system is ignored). See Figure 3 for the results.

6 CONCLUSIONS

The model is designed for a layer structure of n layers consisting of orthotropic media, with a halfspace as the lowest layer. The wave propagation in soil is modelled using a Fourier Transformation, yielding a polynomial operator. The general solution within one layer is calculated by means of Linear Algebra, and back Fourier transformed to handle the boundary conditions. Some additional considerations allow the case of the loaded layer to be handled; the other special cases - top and semi-infinite bottom layer - can be easily handled as well. The balances

at the layer boundaries deliver equations for the displacement vectors at the layer boundaries. Thus, the stress and displacement vectors can be calculated at arbitrary points in the soil. The model presents a more realistic approach, as it incorporates the orthotropic behavior of soils. It showed high accuracy in algebraic as well as in physical tests. Moreover, the calculations are merely decoupled, so the model is excellently suitable for parallelization.
The problems the model has to deal with are merely caused by the fact that the soil parameters are in general not determined well enough. Empiric tests as well as future research will focus on that, promising attempts have already been made[6] and can probably be applied.

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References

[1] H. Hung and Y. Yang, ”A Review of Researches on Ground-Borne Vibrations with Emphasis on Those Induced by Trains”, in *Proceedings of the National Science Council ROC(A)*, Vol. 24(1), (2001), pp. 1–16.

[2] J. Lothe, ”Body Waves in Anisotropic Elastic Media”, in *Modern Theory of Anisotropic Elasticity and Applications*, edited by J. J. Wu, T. C. T. Ting, ,D. M. Barnett, (Society of Industrial and Applied Mathematics, Philadelphia 1990), pp. 173–185.

[3] F. Ziegler and P. Borejko, ”Seismic Waves in Layered Soil: The Generalized Ray Theory”, in *Structural Dynamics-Recent Advances*, edited by G. I. Schueller, (Springer, Berlin, 1991), pp. 52–90.

[4] I. Tsvankin, ”Seismic Signatures and Analysis of Reflection data in Anisotropic media”, volume 29 of *Handbook of Geophysical Exploration: Seismic Exploration*, (Elsevier science, Amsterdam, 2001).

[5] H. Waubke, ”Waves in a Layered Orthotropic Medium with and without Random Properties”, in *10th International Congress on Sound and Vibration*, (Stockholm, 2003).

[6] H. Waubke, P. Balazs, B. Jilge and W. Kreuzer, ”Waves in Random Layers with Arbitrary Covariance Functions”, in *12th International Congress on Sound and Vibration*, (Lisbon,2005).