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INVESTIGATION OF A COUPLED FINITE ELEMENT – WAVE BASED TECHNIQUE FOR 2D STEADY-STATE ACOUSTIC ANALYSIS COMPARING A DIRECT AND INDIRECT APPROACH

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Abstract

The finite element method (FEM) is the most commonly used prediction method for steady-state acoustic analysis. However, due to the application of approximating shape functions, the FEM is restricted to problems in the low-frequency range. A recently developed wave based technique (WBT) has proven to be a computationally more efficient prediction method, as compared to the FEM. As a result, the WBT is able to tackle problems at higher frequencies. However, in order to fully benefit from the WBT's computational efficiency, the considered problem should have a moderate geometrical complexity. To overcome the limitations of both prediction methods, hybrid methods have been developed which couple the FEM and the WBT. These hybrid approaches combine the flexibility of the FEM for modelling problems of arbitrary geometry with the computational efficiency of WBT. This paper focuses on a direct hybrid coupling approach between the FEM and the WBT. The performance of the hybrid approach is compared with that of the FEM by means of a two-dimensional validation example.

INTRODUCTION

The finite element method (FEM) is the most commonly used deterministic prediction method for the analysis of steady-state acoustic problems [1]. Due to a

discretization of the acoustic problem domain into (small) elements, the FEM can tackle problems of arbitrary geometrical complexity. However, the FEM is restricted to problems in the low-frequency range due to the increasing model size and computational efforts with increasing frequency in order to keep the FE approximation errors within reasonable bounds.
The recently developed wave based technique (WBT) has proven to be a computationally more efficient deterministic prediction method, as compared to the FEM [2]. The WBT is based on an indirect Trefftz approach, in that the approximation solution satisfies the governing domain equations and violates only the boundary conditions. The resulting numerical models are much smaller than corresponding FE models. Combined with a superior convergence rate, the small model size makes the WBT applicable for analyses at higher frequencies as compared to the low-frequency application range of the FEM. However, in order to fully exploit the WBT's computational efficiency, the problem at hand should have a moderate geometrical complexity.
To overcome this drawback of the WBT, hybrid couplings between the FEM and the WBT have been developed. These WBT-HFEMs combine the flexibility of the FEM for modelling problems of arbitrary geometry with the computational efficiency of the WBT. Whereas an indirect hybrid coupling strategy using an auxiliary frame between the FE and WBT subdomains has been described in [3, 4], this paper focuses on a direct hybrid coupling approach between the FEM and the WBT. A two-dimensional validation example compares the performance of the WBT-HFEMs with the FEM.

PROBLEM DEFINITON

Consider a two-dimensional (2D) interior steady-state acoustic problem involving a 2D acoustic cavity Ω surrounded by a boundary Γ. The homogeneous Helmholtz equation

$$\left(\Delta + k^2\right) p(\mathbf{r}) = 0, \quad \forall \mathbf{r} \in \Omega \tag{1}$$

governs the steady-state pressure $p(\mathbf{r}) = p(x, y)$ in Ω, with $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ the Laplace operator, $k = \omega/c$ the acoustic wave number, ω the circular frequency, ρ the ambient fluid density and $j = \sqrt{-1}$ the unit imaginary number. The boundary Γ consists of three parts $\left(\Gamma = \Gamma_p \cup \Gamma_v \cup \Gamma_Z\right)$, where the following boundary conditions are imposed

$$p = \overline{p} \ \text{ at } \Gamma_p, \ \frac{j}{\rho\omega}\frac{\partial p}{\partial n} = \overline{v}_n \ \text{ at } \Gamma_v \ \text{ and } \frac{j}{\rho\omega}\frac{\partial p}{\partial n} = \frac{p}{\overline{Z}} \ \text{ at } \Gamma_Z \tag{2}$$

$\overline{p}$, $\overline{v}_n$ and $\overline{Z}$ represent the prescribed pressure, the prescribed normal velocity and the prescribed impedance, respectively.

HYBRID WBT – FEM THEORY

Review of the finite element method

The FEM discretizes the acoustic cavity Ω into a large number of non-overlapping elements. Each element Ω^e is surrounded by the element boundary Γ^e. This boundary is composed of four parts $\left(\Gamma^e = \Gamma_p^e \cup \Gamma_v^e \cup \Gamma_Z^e \cup \Gamma_i^e\right)$, which are the intersections of Γ^e with the problem boundaries ($\Gamma_p^e = \Gamma^e \cap \Gamma_p$, $\Gamma_v^e = \Gamma^e \cap \Gamma_v$, $\Gamma_Z^e = \Gamma^e \cap \Gamma_Z$) and the common interface Γ_i^e between two adjacent elements.
A linear combination of simple (polynomial) shape functions N_a approximates the exact solution p within each element Ω^e as follows

$$p(\mathbf{r}) \approx \hat{p}(\mathbf{r}) = \sum_{a=1}^{n_a} N_a(\mathbf{r})\, p_a^e = \mathbf{N}(\mathbf{r})\mathbf{p}^e, \quad \forall \mathbf{r} \in \Omega^e \tag{3}$$

The contribution factors p_a^e, stored in the element vector $\mathbf{p}^e$, form the unknown degrees of freedom (DOF's). Approximation $\hat{p}$ satisfies a priori the essential boundary conditions along Γ_p^e and the inter-element pressure continuity between two adjacent elements along Γ_i^e. Errors on the Helmholtz equation (1) and the mixed and natural boundary conditions (2) are forced to zero by a weighted residual formulation. Application of partial integration and the divergence theorem, transforms the weighted residual formulation into its weak form.

$$\int_{\Omega^e} \left(-(\nabla W)^T (\nabla \hat{p}) + k^2 W \hat{p}\right) d\Omega - \int_{\Gamma_v^e} j\rho\omega W \overline{v}_n d\Gamma - \int_{\Gamma_Z^e} j\rho\omega W \frac{\hat{p}}{\overline{Z}} d\Gamma - \int_{\Gamma_i^e} j\rho\omega W v_i^e d\Gamma = 0 \tag{4}$$

W represents a weighting function and v_i^e the unknown interface velocity. Following a Galerkin approach, the weighting functions are chosen to be a linear combination of the same basis functions N_a as used for the pressure approximation (3).

In an assembly of all element models into one global FE model the terms resulting from inter-element velocity continuity on Γ_i^e cancel out each other. Substitution of the approximation expansion (3) and the weighting function expansion into the weak form of the weighted residual formulation (4) yields the following FE model

$$\left(-\omega^2\mathbf{M} + j\omega\mathbf{C} + \mathbf{K}\right)\mathbf{p} = -j\omega\mathbf{v}_n \tag{5}$$

with $\mathbf{M} = \int\limits_{\Omega} \frac{1}{c^2}\mathbf{N}^{\mathrm{T}}\mathbf{N}d\Omega$ $\quad$ $\mathbf{C} = \int\limits_{\Gamma_Z} \rho \frac{1}{\overline{Z}}\mathbf{N}^{\mathrm{T}}\mathbf{N}d\Gamma$

$$\mathbf{K} = \int\limits_{\Omega} (\nabla\mathbf{N})^{\mathrm{T}}(\nabla\mathbf{N})d\Omega \quad \text{and} \quad \mathbf{v}_n = \int\limits_{\Gamma_v} \rho\mathbf{N}^T\overline{v}_n d\Gamma .$$

Review of the wave based technique

In the WBT the problem domain Ω is partitioned into a small number of (large) convex subdomains. This technique expresses the pressure field in each subdomain as a linear combination of Trefftz basis functions Φ_a, which satisfy the homogeneous Helmholtz equation within the subdomain, but which may violate the boundary conditions [2].

$$p(\mathbf{r}) \approx \hat{p}(\mathbf{r}) = \sum_{a=1}^{n_a} \Phi_a(\mathbf{r})q_a = \mathbf{\Phi}(\mathbf{r})\mathbf{q}, \quad \forall\mathbf{r}\in\Omega \tag{6}$$

The DOF's are the unknown contribution factors q_a, which are stored in $\mathbf{q}$. The Trefftz basis functions Φ_a are selected from the following set of propagating and evanescent wave functions

$$\begin{aligned} \Phi_r(\mathbf{r}) &= \cos(k_{rx}x)e^{-jk_{ry}y} \\ \Phi_s(\mathbf{r}) &= e^{-jk_{sx}x}\cos(k_{sy}y) \end{aligned} \quad \text{with} \quad \begin{aligned} k_{rx} &= \frac{r\pi}{L_x},\ k_{ry} = \pm\sqrt{k^2 - k_{rx}^2}, \quad \text{for } r = 0,1,\dots,n_r \\ k_{sy} &= \frac{s\pi}{L_y},\ k_{sx} = \pm\sqrt{k^2 - k_{sx}^2}, \quad \text{for } s = 0,1,\dots,n_s \end{aligned} \tag{7}$$

where the wave number components depend on the dimensions L_x and L_y of a rectangular bounding box, which encloses the considered subdomain Ω. The integer numbers n_r and n_s are limited by the following truncation rule

$$n_r = \left\lceil T\frac{kL_x}{\pi} \right\rceil \quad \text{and} \quad n_s = \left\lceil T\frac{kL_y}{\pi} \right\rceil \tag{8}$$

with a user defined truncation parameter T.
The approximation errors at the boundaries are enforced to zero in an integral sense applying the following weighted residual formulation

$$-\int\limits_{\Gamma_p} \frac{j}{\rho\omega}\frac{\partial W}{\partial n}(\hat{p} - \overline{p})d\Gamma + \int\limits_{\Gamma_v} W\left(\frac{j}{\rho\omega}\frac{\partial\hat{p}}{\partial n} - \overline{v}_n\right)d\Gamma + \int\limits_{\Gamma_Z} W\left(\frac{j}{\rho\omega}\frac{\partial\hat{p}}{\partial n} - \frac{\hat{p}}{\overline{Z}}\right)d\Gamma = 0 \tag{9}$$

where W is a weighting function. Following a Galerkin approach, W is chosen to be a linear combination of the same Trefftz basis functions Φ_a (7) as used for the pressure approximation (6).

Substitution of the pressure approximation (6) and the weighting function expansion in the weighted residual formulation (9) results in the following WB model

$$\left(\mathbf{A}_p + \mathbf{A}_v + \mathbf{A}_Z\right) \cdot \mathbf{q} = \mathbf{b}_p + \mathbf{b}_v \tag{10}$$

with $\mathbf{A}_p = -\int_{\Gamma_p} \frac{j}{\rho\omega} \frac{\partial \mathbf{\Phi}^T}{\partial n} \mathbf{\Phi} d\Gamma$, $\mathbf{A}_v = \int_{\Gamma_p} \mathbf{\Phi}^T \frac{j}{\rho\omega} \frac{\partial \mathbf{\Phi}}{\partial n} d\Gamma$, $\mathbf{A}_Z = \int_{\Gamma_Z} \mathbf{\Phi}^T \left(\frac{j}{\rho\omega} \frac{\partial \mathbf{\Phi}}{\partial n} - \frac{\mathbf{\Phi}}{\overline{Z}} \right) d\Gamma$

$\mathbf{b}_p = -\int_{\Gamma_p} \frac{j}{\rho\omega} \frac{\partial \mathbf{\Phi}^T}{\partial n} \overline{p} d\Gamma$, $\mathbf{b}_v = \int_{\Gamma_p} \mathbf{\Phi}^T \overline{v}_n d\Gamma$.

Hybrid Finite Element – Wave Based Technique

In order to combine the strengths of the FEM and the WBT, a hybrid coupling approach is proposed. The key idea of such a hybrid FE-WB approach is to model large, homogeneous, geometrically simple subdomains with the WBT, while the FEM is employed to model the geometrically more complex regions. As a result, the considered hybrid models contain less DOF's than the equivalent pure FE models. At the hybrid interface Γ_i between the regions modeled with the FEM and those modeled with the WBT, pressure continuity and normal velocity continuity conditions are weakly enforced. At subdomain level, the acoustic problems are well-posed if at each subdomain boundary point one boundary condition is imposed. In order to comply with this requirement, pressure continuity is imposed as a boundary condition on the associated WBT subdomains along Γ_i and normal velocity continuity is imposed as a boundary condition on the associated FE subdomains along Γ_i.

$$\frac{j}{\rho\omega} \frac{\partial p_{FE}}{\partial n_{FE}} = -\frac{j}{\rho\omega} \frac{\partial p_{WB}}{\partial n_{WB}} \quad \text{on } \Gamma_i \quad \text{and} \quad p_{WB} = p_{FE} \quad \text{on } \Gamma_i \tag{11}$$

To impose these continuity conditions in an indirect coupling approach [3,4], an auxiliary frame with associated frame DOF's is introduced. In a direct coupling approach, no additional DOF's are needed, and the continuity conditions are imposed directly on the approximation fields of the FEM and the WBT.
Along the hybrid interface Γ_i, velocity continuity is imposed. This results in an additional term in the weighted residual formulation (4)

$$\cdots - \int_{\Gamma_i} W \frac{\partial \hat{p}_{WB}}{\partial n_{WB}} d\Gamma \tag{12}$$

The novel weighted residual formulation leads to the following FE part of the hybrid model

$$\left(-\omega^2\mathbf{M^1} + j\omega\mathbf{C^1} + \mathbf{K^1}\right)\mathbf{p_1} + \mathbf{Q_{12}q_2} = -j\omega\mathbf{v_{n1}} \tag{13}$$

with

$$\mathbf{Q_{12}} = \int_{\Gamma_i} \mathbf{N^T}\frac{\partial\mathbf{\Phi}}{\partial n_{WB}}d\Gamma \tag{14}$$

where $\mathbf{Q_{12}}$ represents the coupling matrix between FE and WB.

To impose pressure continuity along the interface Γ_i, the weighted residual formulation (9) of the WB part of the hybrid model is extended with the following term

$$\cdots - \int_{\Gamma_i} \frac{j}{\rho\omega}\frac{\partial W}{\partial n_{WB}}\left(\hat{p}_{WB} - \hat{p}_{FE}\right)d\Gamma \tag{15}$$

This modified weighted residual formulation results in the following WB part of the hybrid model

$$\left(\mathbf{A}_{p2} + \mathbf{A}_{v2} + \mathbf{A}_{Z2} + \mathbf{C}_2\right)\cdot\mathbf{q}_2 + \mathbf{Q}_{21}\cdot\mathbf{p}_1 = \mathbf{b}_{p2} + \mathbf{b}_{v2} \tag{16}$$

with

$$\mathbf{Q}_{21} = \int_{\Gamma_i} \frac{j}{\rho\omega}\frac{\partial\mathbf{\Phi}}{\partial n_{WB}}^T \mathbf{N}d\Gamma \quad \text{and} \quad \mathbf{C}_2 = -\int_{\Gamma_i} \frac{j}{\rho\omega}\frac{\partial\mathbf{\Phi}}{\partial n_{WB}}^T \mathbf{\Phi}d\Gamma \tag{17}$$

where $\mathbf{Q_{21}}$ represents the coupling matrix between WB and FE and $\mathbf{C}_2$ represents the WB model back-coupling matrix. The complete hybrid FE-WB model is given by

$$\begin{bmatrix} -\omega^2\cdot\mathbf{M^1} + j\omega\cdot\mathbf{C^1} + \mathbf{K^1} & \mathbf{Q}_{12} \\ \mathbf{Q}_{21} & \mathbf{A}_{p2} + \mathbf{A}_{v2} + \mathbf{A}_{Z2} + \mathbf{C}_2 \end{bmatrix}\begin{Bmatrix} \mathbf{p}_1 \\ \mathbf{q}_2 \end{Bmatrix} = \begin{Bmatrix} -j\omega\mathbf{v}_{n1} \\ \mathbf{b}_2 \end{Bmatrix}. \tag{18}$$

Solution of (18) for the DOF's yields the nodal FE pressures $\mathbf{p}_1$ and the WBT wave function contribution factors $\mathbf{q}_2$.

NUMERICAL VALIDATION EXAMPLES

In order to validate the novel direct hybrid method, a simple bounded 2D acoustic problem is considered, see figure 1. A normal velocity excitation $\left(\overline{v}_n = 1[m/s]\right)$ and normal impedance boundary conditions $\left(\overline{Z} = 2000[Pa\cdot s/m]\right)$ are imposed. Three types of models are considered: (i) pure FE models with linear quadrilateral elements, (ii) hybrid FE-WB models, which are derived from the pure FE models by replacing a large number of FE with a small number of WB subdomains, coupled in a direct way, or (iii) coupled in an indirect way. A very accurate FE model is used as reference model (34861 DOF's, element length = 0.005m). The model has dimension of $L_x = 1.6[m]$, $L_y = 0.7[m]$ and is filled with air ($\rho = 1.225[kg/m^3]$, $c = 340[m/s]$).

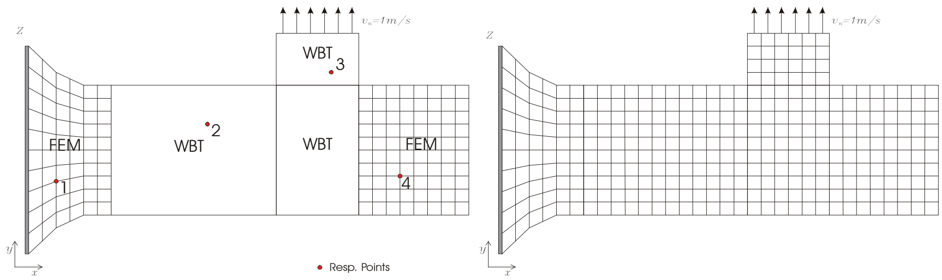

Figure 1 – hybrid FE-WB model (left) and finite element model (right)

Figure 2 plots the pressure amplitude spectrum predictions at point 2 inside the acoustic cavity (see Fig 1). Pressure predictions are shown for a FE model (with 12661 DOF's), an indirect coupled hybrid FE-WB (with 5246 FE DOF's, 120 frame DOF's and 74 WB DOF's at 1000Hz) model and a direct hybrid FE-WB model (with 5246 FE DOF's and 74 WB DOF's at 1000Hz) in a frequency range from 1Hz to 1000Hz with a resolution of 1Hz. The results show no significant differences between the hybrid coupling methods and FEM.

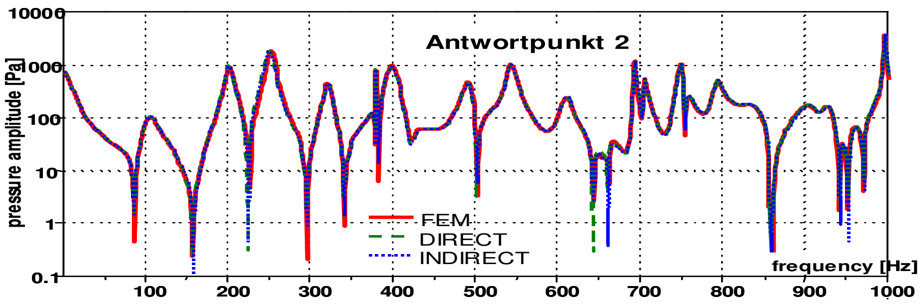

Figure 2 – pressure spectrum prediction for 2D acoustic problem

Figure 3 shows results from a convergence analysis at 1560Hz, plotting the average relative pressure prediction accuracy using 391 evenly distributed response points in function of the number of DOF's and the involved CPU time. The refinement of the hybrid FE-WB models consists of a mesh refinement of the FE submodel only. The WB submodels have a fixed size of 108 wave functions. The various model sizes are listed in table 1. For the reference model the FE model with 34861 DOF's was used (FE length = 0.005m).

Table 1-number of DOF's used for different models

	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6
FEM	391	1449	3235	5689	8831	12661
Direct	284	738	1472	2486	3780	5354
Indirect	304	778	1532	2566	3880	5474

The convergence analysis shows that convergence rates of the direct and the indirect hybrid methods are very similar and higher than that of the FEM.

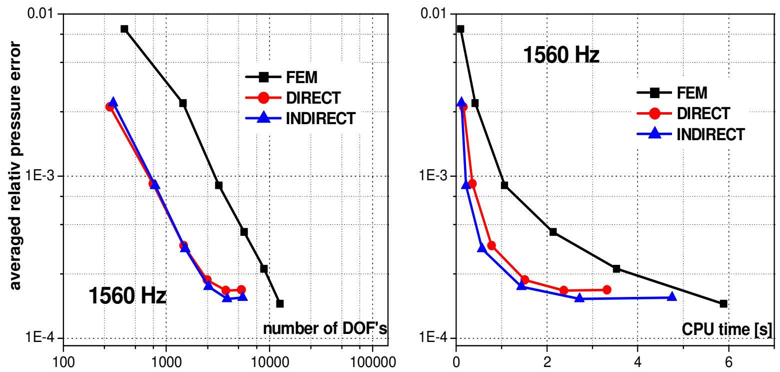

Figure 3 – averaged relative pressure error

CONCLUSIONS AND NEXT STEPS

This paper discusses a direct hybrid coupling approach between the FEM and the WBT as an alternative for the indirect coupling approach discussed in [3,4]. A two-dimensional validation example shows that the direct and indirect approach both yield prediction results of similar accuracy. Both approaches exhibit a similar enhanced convergence behavior as compared to the FEM. The direct approach, discussed in this paper, results in smaller numerical models, as compared to the indirect approach. As a result the direct method is less memory demanding than the indirect method. In a next phase, the hybrid FE-WBT method will be implemented and validated for three-dimensional engineering problems.

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