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Fast and accurate solution of the Helmholtz equation

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Abstract

We discuss efficient numerical methods for the boundary integral formulation of various three dimensional boundary value problems for the Helmholtz equation. The corresponding boundary integral equations will be discretised using the Galerkin method leading to a system of linear equations with a dense matrix A of some dimension N. A naive strategy for the solution of the corresponding linear systems would need at least $\mathcal{O}(N^2)$ arithmetical operations and memory. In contrast, the Adaptive Cross Approximation method (ACA) generates the low-rank approximant of the matrix by the use of only few matrix entries leading to almost linear complexity. The efficiency and convergence properties of the numerical method (Galerkin discretisation, ACA approximation of matrices, iterative solution) will be illustrated for a number of different boundary value problems and for various surfaces.

INTRODUCTION

The solutions of second order partial differential equations can be described by certain surface and volume potentials when a fundamental solution of the underlying partial differential equation is known. Although the existence of such a fundamental solution can be guaranteed for a wide class of partial differential operators, see for example [1], the explicit construction of fundamental solutions is possible only for partial differential equations with constant coefficients. In particular, the Helmholtz equation can be solved by the use of boundary integral equations and boundary element methods (BEM). The reduced wave equation or the Helmholtz equation in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ with the boundary Γ is

$$-\Delta u(x) - \kappa^2 u(x) \;=\; 0 \quad \text{for } x \in \Omega\,, \tag{1}$$

where $\kappa > 0$ is the wave number. The solution of (1) is given by the representation formula

$$u(x) \;=\; \int\limits_{\Gamma} u^*_\kappa(x,y)\gamma_1^{\text{int}} u(y) ds_y - \int\limits_{\Gamma} \gamma_{1,y}^{\text{int}} u^*_\kappa(x,y)\gamma_0^{\text{int}} u(y) ds_y \tag{2}$$

for $x \in \Omega$. In (2), the fundamental solution of the Helmholtz equation (1)

$$u^*_\kappa(x,y) \;=\; \frac{1}{4\pi}\frac{e^{\imath\,\kappa|x-y|}}{|x-y|} \quad \text{for } x,y \in \mathbb{R}^3. \tag{3}$$

has been used. The symbol γ_0^{int} denotes the interior trace operator while γ_1^{int} is the interior conormal derivative. Now we define the boundary integral operators. The single layer potential operator $V_\kappa \;:\; H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$, is

$$(V_\kappa w)(x) = \frac{1}{4\,\pi}\int\limits_\Gamma \frac{e^{\imath\,\kappa|x-y|}}{|x-y|} w(y) ds_y \quad \text{for } x \in \Gamma\,.$$

If Γ is a Lipschitz boundary, the operator $V_\kappa - V_0 \;:\; H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is compact. Since the single layer potential V_0 of the Laplace operator is $H^{-1/2}(\Gamma)$–elliptic, the single layer potential V_κ is coercive, i.e. with the compact operator $C = V_0 - V_\kappa$, the Gårdings inequality

$$\langle (V_k + C)w, w\rangle_\Gamma \;=\; \langle V_0 w, w\rangle_\Gamma \;\geq\; c_1^{V_0}\,\|w\|^2_{H^{-1/2}(\Gamma)} \quad \text{for all } w \in H^{-1/2}(\Gamma)$$

is satisfied. The double layer potential operator $K_\kappa \;:\; H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is

$$(K_\kappa v)(x) = \lim_{\varepsilon\to 0}\frac{1}{4\pi}\int\limits_{y\in\Gamma:|y-x|\geq\varepsilon} \Big(\nabla_y \frac{e^{\imath\,\kappa|x-y|}}{|x-y|}, \underline{n}(y)\Big) v(y) ds_y \quad \text{for } x \in \Gamma\,.$$

Its adjoint operator $K'_\kappa \;:\; H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is given by

$$(K'_\kappa w)(x) = \lim_{\varepsilon\to 0}\frac{1}{4\pi}\int\limits_{y\in\Gamma:|y-x|\geq\varepsilon} \Big(\nabla_x \frac{e^{\imath\,\kappa|x-y|}}{|x-y|}, \underline{n}(x)\Big) w(y) ds_y\,.$$

The conormal derivative of the double layer potential defines the hypersingular boundary integral operator $D_\kappa \;:\; H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$. For a Lipschitz boundary Γ, the modified operator $D_\kappa - D_0 \;:\; H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is compact. Since the regularised hypersingular boundary integral operator $D_0 + I$ of the Laplace operator is $H^{1/2}(\Gamma)$–elliptic, and since the embedding $H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is compact, the hypersingular boundary integral operator D_κ is coercive, i.e. with the compact operator $C = D_\kappa - D_0 - I$, the Gårding's inequality

$$\langle (D_\kappa + C)v, v\rangle_\Gamma \;=\; \langle (D_0 + I)v, v\rangle_\Gamma \;\geq\; c_1^{\widetilde{D}_0}\,\|v\|^2_{H^{1/2}(\Gamma)}$$

is satisfied for all $v \in H^{1/2}(\Gamma)$. The bilinear form for the hypersingular boundary integral for the Helmholtz equation can be written in the following form, see [2]:

$$\begin{aligned}\int\limits_\Gamma (D_\kappa u)(x) v(x) ds_x \;&=\; \frac{1}{4\pi}\int\limits_\Gamma\int\limits_\Gamma \frac{e^{\imath\,\kappa|x-y|}}{|x-y|}\big(\underline{\text{curl}}_\Gamma u(y), \underline{\text{curl}}_\Gamma v(x)\big) ds_y ds_x \\ &\quad -\kappa^2\int\limits_\Gamma\int\limits_\Gamma \frac{e^{\imath\,\kappa|x-y|}}{|x-y|} u(y)v(x)\big(\underline{n}(x), \underline{n}(y)\big) ds_y ds_x\,. \end{aligned} \tag{4}$$

In addition, we also consider the exterior boundary value problem

$$-\Delta u(x) - \kappa^2 u(x) \;=\; 0 \quad \text{for } x \in \Omega^e = \mathbb{R}^3 \backslash \overline{\Omega}\,, \tag{5}$$

where we have to add the Sommerfeld radiation condition

$$\left| \Big(\frac{x}{|x|}, \nabla u(x) \Big) - \imath\, \kappa u(x) \right| \;=\; \mathcal{O}\left(\frac{1}{|x|^2} \right) \quad \text{as } |x| \to \infty. \tag{6}$$

BOUNDARY VALUE PROBLEMS

We first consider the interior Dirichlet boundary value problem for the Helmholtz equation

$$-\Delta u(x) - \kappa^2 u(x) \;=\; 0 \quad \text{for } x \in \Omega, \quad \gamma_0^{\text{int}} u(x) \;=\; g(x) \quad \text{for } x \in \Gamma\,. \tag{7}$$

Using the representation formula (2), the solution of the above problem is given by

$$u(x) \;=\; \int_\Gamma u_\kappa^*(x,y) t(y) ds_y - \int_\Gamma \gamma_{1,y}^{\text{int}} u^*(x,y) g(y) ds_y \quad \text{for } x \in \Omega,$$

where $t = \gamma_1^{\text{int}} u \in H^{-1/2}(\Gamma)$ is the unknown conormal derivative of u on Γ which has to be determined from the variational problem

$$\Big\langle V_\kappa t, w \Big\rangle_\Gamma \;=\; \Big\langle \Big(\frac{1}{2} I + K_\kappa \Big) g, w \Big\rangle_\Gamma \quad \text{for all } w \in H^{-1/2}(\Gamma). \tag{8}$$

The boundary integral operator V_κ is singular, and, therefore, not invertible, if $\kappa^2 = \lambda$ is an eigenvalue of the Dirichlet eigenvalue problem for the Laplace equation. On the other hand, if κ^2 is not an eigenvalue of the Dirichlet eigenvalue problem, the single layer potential V_κ is injective and hence, since V_κ is coercive, also invertible. In addition, we consider the exterior Neumann boundary value problem

$$-\Delta u(x) - \kappa^2 u(x) \;=\; 0 \quad \text{for } x \in \Omega^e, \quad \gamma_1^{\text{int}} u(x) \;=\; g(x) \quad \text{for } x \in \Gamma, \tag{9}$$

where we have to require the Sommerfeld radiation condition (6). Due to the radiation condition, the exterior Neumann boundary value problem is uniquely solvable. The solution of the above boundary value problem is given by the representation formula

$$u(x) \;=\; -\int_\Gamma u_\kappa^*(x,y) g(y) ds_y + \int_\Gamma \gamma_{1,y}^{\text{ext}} u_\kappa^*(x,y) \gamma_0^{\text{ext}} u(y) ds_y \quad \text{for } x \in \Omega^e.$$

The unknown Dirichlet datum $\overline{u} = \gamma_0^{\text{ext}} u$ has to be found from the variational problem

$$\Big\langle D_\kappa \bar{u}, v \Big\rangle_\Gamma \;=\; -\Big\langle \Big(\frac{1}{2} I + K_\kappa' \Big) g, v \Big\rangle_\Gamma \quad \text{for all } v \in H^{1/2}(\Gamma)\,. \tag{10}$$

Since the hypersingular boundary integral operator D_κ of the exterior Neumann boundary value problem coincides with the operator which is related to the interior Neumann boundary value problem, D_κ is not invertible when κ^2 is an eigenvalue of the interior Neumann eigenvalue problem for the Laplace equation. In fact, the variational problem (10) of the direct approach is solvable, but the solution is not unique; we skip the details. If κ^2 is not an eigenvalue of the Neumann eigenvalue problem, then the unique solvability of the variational problem (10) follows, since D_κ is coercive and injective.

DISCRETISATION

For $N \in \mathbb{N}$, we consider a sequence of boundary element meshes

$$\Gamma_N = \bigcup_{\ell=1}^{N} \overline{\tau}_\ell \,. \tag{11}$$

In the most simplest case, we assume that Γ is piecewise polyhedral and that each boundary element mesh (11) consists of N plane triangular boundary elements τ_ℓ. We assume that all boundary elements τ_ℓ are uniformly shape regular.

The simplest choice of the trial functions are the piecewise constant basis functions

$$\psi_\ell(x) \;=\; \begin{cases} 1 & \text{for } x \in \tau_\ell, \\ 0 & \text{elsewhere} \end{cases} \tag{12}$$

for $\ell = 1, \ldots, N$. The global trial space is $S_h^0(\Gamma) \;=\; \text{span}\{\,\psi_\ell\,\}_{\ell=1}^N, \quad \dim S_h^0(\Gamma) \;=\; N$. The following approximation property holds in $S_h^0(\Gamma)$. Let $w \in H^s_{\text{pw}}(\Gamma)$ for some $s \in [0,1]$. Then it holds

$$\inf_{w_h \in S_h^0(\Gamma)} \|w - w_h\|_{H^\sigma(\Gamma)} \;\le\; c\, h^{s-\sigma}\, |w|_{H^s_{\text{pw}}(\Gamma)} \tag{13}$$

for all $\sigma \in [-1, 0]$. Furthermore, one can define globally continuous piecewise linear basis functions φ_j with

$$\varphi_j(x) \;=\; \begin{cases} 1 & \text{for } x = x_j, \\ 0 & \text{for } x = x_i \neq x_j, \quad , \\ \text{piecewise linear} & \text{elsewhere.} \end{cases}$$

where $x_j, j = 1, \ldots, M$ are the nodes of the discretisation (11). The basis functions φ_j are used to define the trial space $S_h^1(\Gamma) \;=\; \text{span}\{\varphi_j\}_{j=1}^M, \quad \dim S_h^1(\Gamma) \;=\; M$. Let $v \in H^s_{\text{pw}}(\Gamma)$ for some $s \in [1, 2]$. Then there holds

$$\inf_{v_h \in S_h^1(\Gamma)} \|v - v_h\|_{H^\sigma(\Gamma)} \;\le\; c\, h^{s-\sigma}\, |v|_{H^s_{\text{pw}}(\Gamma)} \tag{14}$$

for all $\sigma \in [-2, 1]$. Using a sequence of finite dimensional subspaces $S_h^0(\Gamma)$ spanned by piecewise constant basis functions, approximate solutions

$$t_h \;=\; \sum_{\ell=1}^{N} t_\ell \psi_\ell \in S_h^0(\Gamma)$$

of the interior Dirichlet boundary value problem (7) are obtained from the Galerkin equations

$$\Big\langle V_\kappa t_h, \psi_k \Big\rangle_\Gamma \;=\; \Big\langle \Big(\frac{1}{2} I + K_\kappa\Big) g, \psi_k \Big\rangle_\Gamma \quad \text{for } k = 1, \ldots, N. \tag{15}$$

Hence, we find the coefficient vector $\underline{t} \in \mathbb{C}^N$ as the unique solution of the linear system

$$V_{\kappa,h} \underline{t} \;=\; \underline{f}$$

with

$$V_{\kappa,h}[k,\ell] = \frac{1}{4\pi} \int\limits_{\tau_k} \int\limits_{\tau_\ell} \frac{e^{\imath\kappa|x-y|}}{|x-y|} ds_y ds_x \,, \tag{16}$$

$$f_k = \frac{1}{2} \int\limits_{\tau_k} g(x) ds_x + \frac{1-\imath\,\kappa}{4\pi} \int\limits_{\tau_k} \int\limits_{\Gamma} e^{\imath\,\kappa|x-y|} \frac{(x-y, \underline{n}(y))}{|x-y|^3} g(y) ds_y ds_x$$

for $k, \ell = 1, \ldots, N$. To compute an approximate solution of the exterior Neumann problem (9), we consider the Galerkin equations

$$\Big\langle D_\kappa \widetilde{u}_h, \varphi_i \Big\rangle_\Gamma \;=\; \Big\langle \Big(-\frac{1}{2} I - K'_\kappa\Big) g, \varphi_i \Big\rangle_\Gamma \quad \text{for } i = 1, \ldots, M.$$

Hence, we obtain the coefficient vector $\underline{\widetilde{u}} \in \mathbb{C}^M$ as the unique solution of the linear system

$$D_{\kappa,h} \underline{\widetilde{u}} = \underline{f},$$

with

$$\begin{aligned} D_{\kappa,h}[i,j] \;=\; & \frac{1}{4\pi} \int\limits_{\Gamma} \int\limits_{\Gamma} \frac{e^{\imath\kappa|x-y|}}{|x-y|} (\underline{\mathrm{curl}}_\Gamma \varphi_j(y), \underline{\mathrm{curl}}_\Gamma \varphi_i(x)) ds_y ds_x \\ & -\kappa^2 \int\limits_{\Gamma} \int\limits_{\Gamma} \frac{e^{\imath\kappa|x-y|}}{|x-y|} \varphi_j(y) \varphi_i(x) (\underline{n}(x), \underline{n}(y)) ds_y ds_x \,, \end{aligned} \tag{17}$$

$$f_i = -\frac{1}{2} \int\limits_{\Gamma} g(x) \varphi_i(x) ds_x - \frac{1-\imath\,\kappa}{4\pi} \int\limits_{\Gamma} \varphi_i(x) \int\limits_{\Gamma} e^{\imath\,\kappa|x-y|} \frac{(x-y, \underline{n}(y))}{|x-y|^3} g(y) ds_y ds_x$$

for $i, j = 1, \ldots, M$.

APPROXIMATION OF MATRICES

The matrices involved in BEM are dense, i.e. all their entries do not vanish in general, leading to an asymptotically quadratic memory requirement for the whole procedure. Fortunately, all boundary element matrices can be decomposed into a hierarchical system of blocks which

can be approximated by the use of low rank matrices. The formal definition and description of hierarchical matrices as well as operations involving those matrices can be found in [3, 4] and the description of the Adaptive Cross Approximation in [5, 6]. An application of the ACA to the collocation method for the Helmholtz equation can be found in [7]. Thus, a hierarchical approximation of large dense matrices arising from some generating function having diagonal singularity (cf. (3)) consist of three steps: Construction of clusters for variables x and y, Finding of possible admissible blocks, Low rank approximation of admissible blocks. A pair of clusters (Cl_x, Cl_y) is admissible if

$$\min\Big(\operatorname{diam}(Cl_x), \operatorname{diam}(Cl_y)\Big) \le \eta \operatorname{dist}(Cl_x, Cl_y)\,,$$

where $0 < \eta < 1$ is a given parameter. In Fig. 1 the second level of separation of a simplified model of an exhaust manifold in a system of clusters is shown on left. The middle plot presents an admissible cluster pair which will lead to an admissible block of the system matrix. The

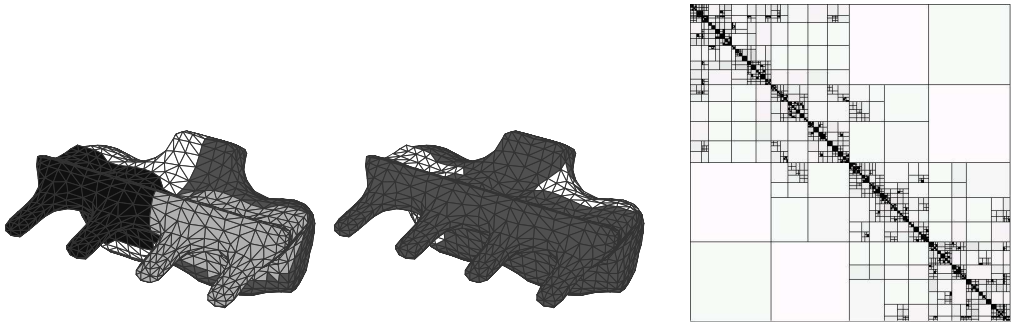

Figure 1: Hierarchical matrix

block structure of the Galerkin matrix is shown on the right. The colour of the blocks indicates the "quality" of the approximation. The light grey colour corresponds to well approximated blocks while dark grey and especially black colour indicates worse approximation or even exact computation. For a block $A \in \mathbb{R}^{n\times m}$, the fully pivoted ACA algorithm can be written in the following form:

Algorithm 1

1. Initialisation $R_0 = A\,,\ \ S_0 = 0\,.$
2. For $i = 0, 1, 2, \ldots$ compute
   - 2.1. pivot element $(k_{i+1}, \ell_{i+1}) = \operatorname{ArgMax} |(R_i)_{k\ell}|\ ,$
   - 2.2. normalising constant $\gamma_{i+1} = \big((R_i)_{k_{i+1}\ell_{i+1}}\big)^{-1}\ ,$
   - 2.3. new vectors $u_{i+1} = \gamma_{i+1} R_i e_{\ell_{i+1}}\,,\ \ v_{i+1} = R_i^\top e_{k_{i+1}},$
   - 2.4. new residuum $R_{i+1} = R_i - u_{i+1} v_{i+1}^\top\,,$
   - 2.5. new approximation $S_{i+1} = S_i + u_{i+1} v_{i+1}^\top\,.$

See [6] for more details.

NUMERICAL EXAMPLES

A simplified model of an exhaust manifold as shown in Fig. 1 will be used for the interior Dirichlet boundary value problem. Its surface contains $N = 2264$ elements. We perform two uniform mesh refinements in order to obtain meshes with $N = 9056$ and $N = 36224$ elements, respectively. The analytical solution is taken in the form (3) with $y = (0, 0, 0.06)^\top$ for $\kappa = 80$. The results of the computations are shown in Tables 1 and 2. The number of boundary

Table 1: ACA approximation of the Galerkin matrices $K_{\kappa,h}$ and $V_{\kappa,h}$

N	M	ε_1	MB($K_{\kappa,h}$)	%	MB($V_{\kappa,h}$)	%
2264	1134	10^{-3}	16.62	42.4	11.82	15.1
9056	4530	10^{-4}	137.86	22.0	97.08	7.8
36224	18114	10^{-5}	1046.80	10.5	696.65	3.5

elements is listed in the first column of these tables. The second column contains the number of nodes, while in the third column of Table 1, the prescribed accuracy for the ACA algorithm for the approximation of both matrices $K_{\kappa,h} \in \mathbb{R}^{N\times M}$ and $V_{\kappa,h} \in \mathbb{R}^{N\times N}$ is given. The fourth column of this table shows the memory requirements in MByte for the approximate double layer potential matrix $K_{\kappa,h}$. The quality of this approximation in percentage of the original matrix is listed in the next column. The corresponding values for the single layer potential matrix $V\kappa, h$ can be seen in the columns six and seven. The third column of Table 2 shows

Table 2: Accuracy of the Galerkin method, Dirichlet problem

N	M	$Iter$	$Error_1$	CF_1	$Error_2$	CF_2
2264	1134	177	$3.10 \cdot 10^{-1}$	-	$8.88 \cdot 10^{-3}$	-
9056	4530	208	$1.40 \cdot 10^{-1}$	2.2	$1.08 \cdot 10^{-3}$	8.2
36224	18114	244	$5.83 \cdot 10^{-2}$	2.4	$9.27 \cdot 10^{-5}$	11.7

the number of iterations required by the GMRES method without preconditioning. The fourth column displays the L_2-error of the Neumann datum, while the next column shows its linear convergence. The last pair of columns of Table 2 shows the absolute error in a prescribed inner point $x^* = (0.145303, 0.1, -0.05)^\top$. Finally, the last column of this table indicates the cubic (or even better) convergence of this quantity. The surface of the unit sphere will be used for the exterior Neumann boundary value problem. As an appropriate discretisation of Γ, we consider the icosahedron that is uniformly triangulated before being projected onto the circumscribed unit sphere. On this way we obtain a sequence $\{\Gamma_N\}$ of almost uniform meshes. The analytical solution is taken in the form (3) for $y = (0.9, 0, 0)^\top \in \Omega$, i.e. close to the boundary of the domain Ω and $\kappa = 4$. The results of the computations in Tables 3 and 4. Note that the two last columns of Table 3 show the results of the approximation of an additional matrix $C_{\kappa,h}$ required while using the hypersingular operator, i.e. the second summand in (17). In Table 4, the accuracy obtained for the whole numerical procedure is presented and the numbers in this table have the same meaning. Note that the convergence of the Galerkin

Table 3: ACA approximation of the Galerkin matrices $K_{\kappa,h}, V_{\kappa,h}$, and $C_{\kappa,h}$

N	M	ε_1	MB($K_{\kappa,h}$)	%	MB($V_{\kappa,h}$)	%	MB($C_{\kappa,h}$)	%
80	42	10^{-2}	0.05	100.0	0.05	50.6	0.01	51.2
320	162	10^{-3}	0.75	94.5	0.64	40.8	0.20	50.3
1280	642	10^{-4}	6.88	54.9	5.66	22.7	2.41	38.3
5120	2562	10^{-5}	53.85	26.9	42.91	10.7	19.17	19.1
20480	10242	10^{-6}	379.09	11.8	302.35	4.72	135.47	8.46

Table 4: Accuracy of the Galerkin method, Neumann problem

N	M	$Iter$	$Error_1$	CF_1	$Error_2$	CF_2
80	42	16	$6.78 \cdot 10^{-1}$	–	$3.24 \cdot 10^{-2}$	–
320	162	23	$1.91 \cdot 10^{-1}$	3.55	$4.99 \cdot 10^{-3}$	6.49
1280	642	31	$5.81 \cdot 10^{-2}$	3.29	$1.24 \cdot 10^{-3}$	4.02
5120	2562	44	$1.42 \cdot 10^{-2}$	4.09	$2.90 \cdot 10^{-4}$	4.28
20480	10242	62	$3.27 \cdot 10^{-3}$	4.34	$7.16 \cdot 10^{-5}$	4.04

method for the unknown Dirichlet datum in the L_2 norm is, corresponding to the theory, almost quadratic. In the inner point $x^* = (1.1, 0.0, 1.0978)^\top \in \Omega^e$, we now observe quadratic convergence (7th column) instead of the cubic order obtained for the Dirichlet problem (cf. Table 2).

References

[1] L. Hörmander, *The Analysis of Linear Partial Differential Operators I.*, Springer (1983).

[2] J. C. Nédélec, *Acoustic and Electromagnetic Equations.*, Springer (2001).

[3] W. Hackbusch, *A sparse matrix arithmetic based on $\mathcal{H}$-matrices. I. Introduction to $\mathcal{H}$-matrices*, Computing, 62(2), (1999), 89–108.

[4] W. Hackbusch, B. N. Khoromskij, *A sparse $\mathcal{H}$–matrix arithmetic. ii. application to multi–dimensional problems.*, Computing, 64, (2000), 21–47.

[5] M. Bebendorf, *Approximation of boundary element matrices*, Numer. Math., 86(4), (2000), 565–589.

[6] M. Bebendorf, S. Rjasanow, *Adaptive low–rank approximation of collocation matrices.*, Computing, 70(1), (2003), 1–24.

[7] O. von Estorff, S. Rjasanow, M. Stolper, O. Zaleski, *Two efficient methods for a multifrequency solution of the Helmholtz equation*, Comput. Vis. Sci., 8(2–4), (2005), 159–167.