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COMPUTATION OF REAL NORMAL MODES FROM COMPLEX EIGENVECTORS

Ulrich Fuellekrug

Institute of Aeroelasticity, German Aerospace Center, DLR Goettingen
Bunsenstrasse 10, D-37073 Goettingen, Germany
Ulrich.Fuellekrug@.dlr.de

Abstract
A method for the computation of real normal modes from complex eigenvectors is described in this paper. The method is based on the exact eigensystem equations, which commonly cannot be used due the problem of modal truncation. A solution to this problem is achieved by a reduction transformation. The theory of the method is described in detail. With the purpose to demonstrate the applicability, a simulated vibration system is employed. In addition, experimental data from modal identification tests on an aircraft are used. It can be shown that the method is able to transform identified complex eigenvectors into real normal modes with good reliability and accuracy.

INTRODUCTION

Analytical models of large aerospace structures are usually set-up with the Finite Element Method and comprise in most cases only mass and stiffness properties. With the purpose to validate and update the Finite Element models, modal identification tests are performed. The results of modal identification tests are mainly the modal parameters like eigenfrequencies, modal damping values, modal masses, and (complex) mode shapes. Since the damping characteristics are usually not analytically modelled with Finite Elements, the eigenvalue analysis with the analytical Finite Element models delivers undamped eigenfrequencies and real normal modes. For the adequate correlation of experimental and analytical data it is required that the experimental data is in the same format, i.e. that undamped eigenfrequencies and real normal modes are available.

Real normal modes and undamped eigenfrequencies of a tested elastomechanical structure can be obtained in principle on three different ways. The **first way** is to **directly measure** the real normal modes and undamped eigenfrequencies. In this case the Phase Resonance Method has to be applied, see e.g. [1]. The Phase Resonance Method uses harmonic excitation and an adjusted exciter force vector for each mode. If the related eigenfrequency is tuned, the inertia forces are in equilibrium with the elastic forces and the excitation forces compensate for the damping of the structure. Eigenfrequencies and real normal modes can directly be measured. However, the application of the Phase Resonance Method is time consuming and thus concepts for combining the Phase Resonance Method with Phase Separation Techniques have been developed [1].

A **second way** of obtaining real normal modes is to **apply special Phase Separation Techniques** to measured data. In [2] a time domain method is proposed that identifies the transition matrix $[A]$ from free decay vibrations. Next, the matrix product $[M]^{-1}[K]$ is extracted from $[A]$ and the solution of the eigenproblem

$$[M]^{-1}[K]\{\phi\}_r = \omega_r^2 \{\phi\}_r \ , \tag{1}$$

which is equivalent to

$$\left(-\omega_r^2[M]+[K]\right)\{\phi\}_r = \{0\} \tag{2}$$

delivers the undamped eigenfrequencies ω_r and real normal modes $\{\phi\}_r$. In [3] the method is modified and further developed by performing a principal component analysis. A similar way for the determination of real normal modes can be achieved by applying frequency domain Phase Separation Techniques. The method ISSPA (Identification of Structural System Parameters) [4] is also able to estimate the matrix product $[M]^{-1}[K]$ and to compute real normal modes according to eq. (2). In the same way the FDPI (Frequency Domain Direct Parameter Identification) approach [5] can be utilized. Reference [6] proposes this way for the determination of real normal modes. In addition, it is proposed here to estimate the mode shapes from a least squares approximation technique when the eigenvalues λ_r are already known from applying e.g. the polyreference time domain method. If the mode shape coefficients are forced to be purely real during the least squares approximation the result are real normal modes.

A **third way** of obtaining real normal modes is to use a set of complex modes which already exists, because it has been previously identified with a Phase Separation Technique. Then it is required to **compute real normal modes from the complex eigenvalues and eigenvectors**. In this class several methods with different degree of sophistication have been proposed. First, so-called simple methods can be used (see e.g. [7] and [8]), where the eigenvectors are scaled in an appropriate manner and the modulus of each eigenvector component is used. The sign is adjusted to 0^o or 180^o depending on the phase angle of each complex eigenvector component. More elaborated and highly sophisticated methods can be found in the literature [9-14] and are briefly discussed in the following. Reference [9] presents a method to maximise the MAC correlation between original and transformed mode shapes and uses it for the computation of real normal modes. The method is generalized in Ref. [10]. The publication [11] introduces a linear transformation between the modal matrices with complex modes and real normal modes. Paper [12] proposes a method which is based on orthogonality constraints. Reference [13] also deals with a linear transformation from complex modes to real normal modes and additionally considers the background and consequences of non-proportional damping. Reference [14] is a comprehensive work on the topic. In principle, the proposed method computes first matrices $[M]$ and $[K]$ from identified complex modes $\{\psi\}_r$ and then computes the real normal modes $\{\phi\}_r$ from eq. (2).

The present article describes a method that is based on a previous paper of the author [15]. The method uses exact eigensystem equations for the computation from complex eigenvectors to real normal modes and solves the problem of modal truncation by a reduction transformation. The method appears clear and easy to implement. The theory of the method is described in the following section.

THEORETICAL BACKGROUND

Basic Equations for Real Normal Modes

The eigensystem equations of a linear, viscously damped elastomechanical structure are

$$\lambda_r^2[M]\{\psi\}_r + \lambda_r[C]\{\psi\}_r + [K]\{\psi\}_r = \{0\}\,, \qquad (3)$$

with $[M],[C]$ and $[K]$ as the physical mass, damping and stiffness matrices. For vibratory elastomechanical structures with a general damping matrix $[C]$ the eigenvalues λ_r and eigenvectors $\{\psi\}_r$ appear in complex conjugate pairs. In most cases, the eigenvectors $\{\psi\}_r$ cannot be scaled in a way that their components are purely real.

Multiplication of eq. (3) with $[M]^{-1}$ and rearrangement yields

$$\begin{bmatrix}[M]^{-1}[C] & [M]^{-1}[K]\end{bmatrix}\cdot\begin{Bmatrix}\{\psi\}_r \\ \lambda_r\{\psi\}_r\end{Bmatrix} = -\lambda_r^2\{\psi\}_r\,. \qquad (4)$$

This equation holds valid for each pair of eigenvector and eigenvalue and can thus be expanded to include all eigenvectors and eigenvalues. Eq. (4) can then be solved for the mass modified stiffness matrix $[M]^{-1}[K]$.

The eigensolution with the mass modified stiffness matrix

$$[M]^{-1}[K]\{\phi\}_r = \omega_r^2\{\phi\}_r \qquad (5)$$

delivers the desired real normal modes $\{\phi\}_r$ and undamped eigenfrequencies ω_r.

Modal Truncation

The problem which remains to be solved consists in the modal truncation. The modal identification usually delivers an incomplete modal model where not all modes are identified. Or with other words: the number of sensors on the structure, which equals the length of the eigenvectors $\{\psi\}_r$, is often larger than n (the number of identified modes). A solution of this problem can be achieved by a reduction transformation that reduces the number of eigenvector components. For this reduction transformation a matrix $[X]$ can be employed which contains some kind of mode shape information. Applying a singular value decomposition (see e.g. [16]) to the matrix $[X]$ it follows

$$[X] = [T][S][V]^T, \tag{6}$$

where $[T]$ already represents the transformation matrix. For matrix $[X]$ it is proposed here to use the real parts of the complex eigenvectors

$$[X] = \left[Re\{\psi\}_1 \quad Re\{\psi\}_2 \quad \cdots \quad Re\{\psi\}_n \right]. \tag{7}$$

Concerning the scaling of the eigenvectors different variants are possible. They can be scaled in a way that the maximum component is 1 and, if appropriate, subsequently rotated in the complex plane in order to minimize the phase deviation.

Instead of the real parts, it is also possible to e.g. utilize the modulus of the eigenvector components with adjusted signs for assembling matrix $[X]$.

Procedure for Computing Real Normal Modes

Using the transformation matrix $[T]$ defined above, the complex eigenvectors $\{\psi\}_r$ are transformed from physical coordinates to reduced coordinates by

$$\{\tilde{\psi}\}_r = [T]\{\psi\}_r. \tag{8}$$

The solution of the equation set (with eigenvectors $\{\tilde{\psi}\}_r$ in reduced coordinates)

$$\left[[\tilde{M}]^{-1}[\tilde{K}] \quad [\tilde{M}]^{-1}[\tilde{C}] \right] \cdot \begin{bmatrix} \{\tilde{\psi}\}_1 & \{\tilde{\psi}\}_2 & \cdots & \{\tilde{\psi}\}_{2n} \\ \lambda_1\{\tilde{\psi}\}_1 & \lambda_2\{\tilde{\psi}\}_2 & \cdots & \lambda_{2n}\{\tilde{\psi}\}_{2n} \end{bmatrix} = \\ = -\left[\lambda_1^2\{\tilde{\psi}\}_1 \quad \lambda_2^2\{\tilde{\psi}\}_2 \quad \cdots \quad \lambda_{2n}^2\{\tilde{\psi}\}_{2n} \right]. \tag{9}$$

delivers the mass modified stiffness matrix $[\tilde{M}]^{-1}\left[\tilde{K}\right]$. The undamped eigenfrequencies ω_r and real normal modes $\{\phi\}_r$ are obtained from the eigensolution

$$[\tilde{M}]^{-1}[\tilde{K}]\{\tilde{\phi}\}_r = \omega_r^2\{\tilde{\phi}\}_r. \tag{10}$$

Finally, the computed real normal modes are transformed back to physical coordinates by employing the orthogonality of the transformation matrix $[T]$

$$\{\phi\}_r = [T]^T \{\tilde{\phi}\}_r . \tag{11}$$

ANALYTICAL EXAMPLE

In order to check the method it was first applied to an analytical vibration system with 11 degrees of freedom. The damping values were selected in a way that the mode shapes exhibit a high level of complexity. Table 1 shows in the first three columns the damped eigenfrequencies f_r, the damping values ζ_r, and the mean phase deviation (MPD), i.e. the mean phase angle of the complex eigenvector components. The next two columns list the undamped eigenfrequencies f_{0r} and the MPD of the real normal modes (which is of course 0). The last column shows the MAC (Modal Assurance Criterion) correlation between the complex modes and the real normal modes. It can be seen that several complex modes are significantly different from the real normal modes.

The utilization of all 11 complex eigenvectors in the proposed computational procedure delivers the exact values for all real normal modes. Next, several sets with fewer complex modes were used for the computation of real normal modes. The results are listed in Table 2. For a given set of complex modes the maximum error in frequency and the minimum MAC correlation is listed. It shows that the agreement of computed and exact normal modes is very high in all cases, even if only very few complex modes are used. Figure 1 shows as an example for modes 5 and 6 the computed real normal modes $\{\phi\}_r$ (solid lines) and the complex modes $\{\psi\}_r$ (modulus with adjusted signs as dashed lines). Although there is a large difference between the modes, the procedure is able to compute nearly the exact real normal modes (MAC correlation better then 99.9 %).

	Complex Modes			Real Modes		Corr
#	fqr	zetar	MPD	fq0r	MPD	MAC
1	2.743 Hz	1.948 %	9 °	2.737 Hz	0 °	97 %
2	2.947 Hz	2.333 %	10 °	2.954 Hz	0 °	97 %
3	7.361 Hz	4.790 %	35 °	7.245 Hz	0 °	79 %
4	7.666 Hz	6.571 %	38 °	7.808 Hz	0 °	76 %
5	11.744 Hz	4.808 %	34 °	11.474 Hz	0 °	58 %
6	11.766 Hz	13.241 %	35 °	12.135 Hz	0 °	55 %
7	15.133 Hz	18.105 %	28 °	15.006 Hz	0 °	46 %
8	15.262 Hz	5.549 %	29 °	15.626 Hz	0 °	45 %
9	18.540 Hz	22.328 %	24 °	18.489 Hz	0 °	46 %
10	18.798 Hz	6.844 %	21 °	19.321 Hz	0 °	44 %
11	27.943 Hz	15.863 %	22 °	28.577 Hz	0 °	97 %

Table 1 – Data of the analytical vibration system

Used Complex Modes	Max. Error in Frequency	Min. MAC Value
3,4,5,6,7,8,9,10	0.14 %	99.7 %
5,6,7,8,9,10	0.56 %	99.7 %
5,6,7,8	0.38 %	99.7 %
7,8,9,10	0.57 %	99.7 %
5,6	0.22 %	99.9 %
7,8	0.37 %	99.7 %
9,10	0.53 %	99.6 %

Table 2 – Results of real normal mode computation

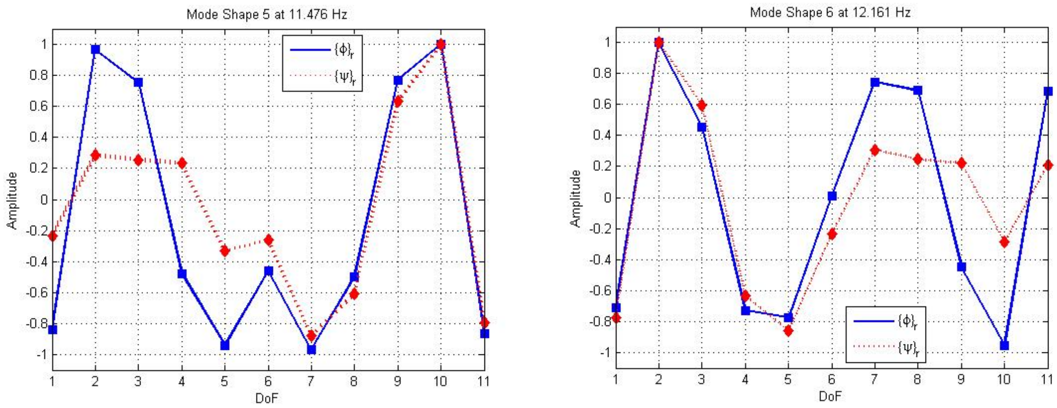

Figure 1 – Computed real normal modes $\{\phi\}_r$ *and complex eigenvectors* $\{\psi\}_r$

EXPERIMENTAL EXAMPLE

The method for computing real normal modes from complex eigenvectors has already been applied to experimental modal data of different aircraft. Figure 2 shows a typical Ground Vibration Test on a sailplane. During the tests usually several configurations with different mass loading are tested The goal is mainly the identification of a complete set of modal parameters up to 60 Hz. For this purpose the Phase Resonance Method is often applied in the first configuration. The other configurations are then tested with a sine sweep excitation and the modes are extracted from the measured FRFs with a frequency domain Phase Separation Technique. The identified complex eigenvectors need then to be transformed to real normal modes.

Table 3 lists, as an example, typical results of a GVT on a sailplane in one configuration. Some complex modes show a rather high mean phase deviation. Here, the complete data set with 22 modes and 80 eigenvector components was transformed from complex to real normal modes in one single computational run. The transformation of the complete data set was possible without any difficulties in this case. However, for other configurations it is sometimes required to split the entire frequency range into two or more parts. In general, it turned out that the method is able to compute real normal modes with good reliability and accuracy.

Figure 2 – Ground Vibration Test on a sailplane

	Complex Modes			Real Modes		Corr
#	fqr	zetar	MPD	fq0r	MPD	MAC
1	3.371 Hz	0.246 %	1 °	3.371 Hz	0 °	100 %
2	5.610 Hz	0.753 %	3 °	5.608 Hz	0 °	100 %
3	7.214 Hz	4.132 %	7 °	7.218 Hz	0 °	100 %
4	8.154 Hz	3.873 %	23 °	8.089 Hz	0 °	98 %
5	8.632 Hz	1.974 %	42 °	8.714 Hz	0 °	81 %
6	11.343 Hz	0.568 %	1 °	11.343 Hz	0 °	100 %
7	13.309 Hz	0.541 %	2 °	13.310 Hz	0 °	100 %
8	14.232 Hz	1.888 %	7 °	14.237 Hz	0 °	100 %
9	15.604 Hz	2.632 %	7 °	15.607 Hz	0 °	100 %
10	20.292 Hz	1.936 %	11 °	20.288 Hz	0 °	98 %
11	21.526 Hz	1.419 %	7 °	21.546 Hz	0 °	99 %
12	22.961 Hz	1.870 %	9 °	22.968 Hz	0 °	99 %
13	25.440 Hz	0.656 %	7 °	25.440 Hz	0 °	100 %
14	26.318 Hz	2.440 %	9 °	26.324 Hz	0 °	99 %
15	29.262 Hz	2.480 %	15 °	29.255 Hz	0 °	98 %
16	31.251 Hz	1.810 %	6 °	31.255 Hz	0 °	100 %
17	32.849 Hz	1.497 %	8 °	32.831 Hz	0 °	99 %
18	36.680 Hz	1.397 %	14 °	36.649 Hz	0 °	99 %
19	38.515 Hz	1.486 %	7 °	38.528 Hz	0 °	99 %
20	39.481 Hz	1.704 %	15 °	39.542 Hz	0 °	98 %
21	44.340 Hz	0.790 %	3 °	44.341 Hz	0 °	100 %
22	58.742 Hz	1.390 %	19 °	58.746 Hz	0 °	100 %

Table 3 – Example for experimental data

SUMMARY AND CONCLUSIONS

The paper first reviews existing methods for the determination of real normal modes. Then a method for the computation of real normal modes from identified complex eigenvectors is described. The theoretical background is explained and the method is illustrated by an analytical example. Test data from a GVT on a sailplane shows the application of the method in practice.

It is intended to implement the method in a software environment which enables to read and write mode shapes in commonly used data formats. In addition and as an alternative, it is planned to also implement some of the quite simple methods for the transformation from complex to real normal modes.

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