ICSV13 - Vienna
The Thirteenth International Congress on Sound and Vibration
Vienna, Austria, July 2-6, 2006

Infinite elements of variable order for exterior acoustic problems

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Abstract

When analyzing exterior acoustic problems employing the finite element method, special measures are needed to satisfy the Sommerfeld radiation condition. Within the last decade infinite elements have become a viable tool for numerical simulations in unbounded media. Up to now, various infinite elements have been developed, where one of the best known formulations are the so-called Astley–Leis elements. Similar to knowledge based concepts, these elements try to resemble the analytic solutions for the Helmholtz equation in an unbounded domain. However, for many problems, such as the sound radiation of slender structures, infinite elements with high approximation orders seem to be necessary only in some local parts of the discretization. Hence, in order to reduce computational costs, it appears reasonable, to vary the order of the infinite elements within the discretization. In this contribution an analysis procedure with infinite elements of variable order is presented. The applicability of common (and eventually modified) error estimators within the analysis process is discussed using representative numerical examples.

INTRODUCTION

Today, the analysis of complex systems with respect to their acoustic behavior is generally achieved by means of numerical methods, such as the finite element method (FEM). A review of developments and recent advances within the field of time-harmonic finite element computations is given in reference [1]. While the FEM is well suited for the investigation of interior acoustic problems, additional effort is necessary, when dealing with unbounded domains. Today, several methods are available to model the sound radiation within finite element computations. The most popular of these approaches are generally referred to as non-reflecting boundary conditions, absorbing layers, and infinite elements. A survey of corresponding methods is given in reference [2] (see, e.g., Huttunen *et al.* [3] or Tezaur *et al.* [4]

for more recent approaches). An advantage of the infinite element approach can be seen in the fact that a solution in the whole unbounded domain is readily obtained within the simulation process. The development of infinite elements started in the 1970s. Regarding exterior acoustic simulations, further developments and generalizations particularly in the last decade have made the combined finite/infinite element approach a reliable simulation tool. Meanwhile, different variants of these elements are also available in various commercial software packages.

The simulations presented in this paper deal with an optimized variant of the so-called Astley–Leis or mapped wave envelope elements. The original formulation of these elements include Lagrange polynomials for the shape approximation in radial direction [5]. For this polynomial basis, however, the resulting system matrix becomes ill-conditioned when used with high approximation orders. Hence, the radial basis functions were later replaced by Legendre polynomials. Motivated by an increasing use of iterative solution algorithms, other concepts have been developed afterwards [6, 7]. The infinite elements employed in the current contribution include Jacobi polynomials for the radial shape approximation. These lead to an improved condition of the overall system matrix and subsequently to more efficient simulations in combination with iterative solution schemes. Investigations regarding the efficiency of these elements are given in reference [8].

The identification of regions with high error within the discretized system is generally accomplished by means of a posteriori error estimators, where an estimate of local errors is computed by processing the numerical solution. These estimators are the basis for analysis procedures where the discretization is locally refined or enriched in order to obtain highly accurate results while saving unneeded degrees of freedom. An overview of commonly employed a posteriori error estimators can be found in reference [9].

Exterior vibro-acoustic investigations often deal with structures that evolve a distinct directionality. Hence, it seems to be appropriate to vary the approximation order of the infinite elements within the discretization. While error estimation for acoustic problems in interior domains (see e.g. [10]) as well as in exterior domains in combination with non-reflecting boundary conditions (see e.g. [11]) have been frequently investigated, the analysis of adaptive strategies for infinite elements remains somewhat unexplored. Quite recently, Demkowicz et al. presented an hp-adaptive method with finite and infinite elements of locally variable order [12]. In their work, however, the adaption of the radial order of an infinite element is linked to the refinement/enrichment of the corresponding underlying finite element. In this contribution, in order to investigate the capabilities of an adaptive enrichment of the infinite elements, the discretization within the conventional elements remains unchanged and the enrichment is restricted to the infinite elements only.

FINITE/INFINITE ELEMENT COMPUTATION

Starting point for the sound radiation analysis is the exterior acoustic problem depicted in Figure 1. Consider a body with a vibrating surface S. The domain of interest is denoted Ω and is separated by the envelope Γ in an interior part Ω_e and an exterior part Ω_i. Inside the

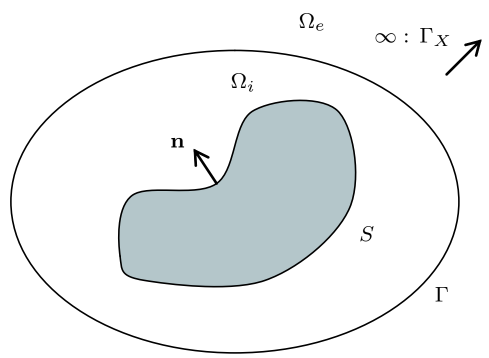

Figure 1: Exterior acoustic problem

acoustic fluid the wave speed c and the fluid density ρ are assumed to be constant, and the sound pressure p is governed by the Helmholtz equation. On the surface S the normal velocity v_n is prescribed by means of Neumann boundary conditions. Together with the Sommerfeld radiation condition on the artificial boundary Γ_X at infinity, the governing boundary value problem reads

$$\Delta p + k^2 p = 0 \qquad \text{in} \quad \Omega \tag{1}$$

$$v_n = \bar{v}_n \qquad \text{on} \quad S \tag{2}$$

$$ikp + \frac{\partial p}{\partial r} = o\left(X^{-(d-1)/2}\right) \qquad \text{on} \quad \Gamma_X \tag{3}$$

where $k = \omega/c$ is the wave number, d is the dimension of the problem and r is the radial direction.

Employing a standard Galerkin weighted residual technique in the interior part and a Petrov–Galerkin scheme in the exterior part leads to the finite/infinite elements formulation of Astley et al. [5]. Hence, conventional finite elements are used within Ω_i and infinite elements are then attached on the envelope in order to fulfill the radiation condition.

The shape functions of the infinite elements may be obtained from a tensor product of the shape approximation on the base (corresponding to the underlying finite element) and a shape function for the radial direction. The radial shape functions are then intentionally chosen, such that with increasing approximation order the analytic solution of the Helmholtz equation is resembled.

The radial basis used in the current work is based on Jacobi polynomials and may be written in a hierarchic form. This is somewhat essential for a local refinement of the element order, since no modification of the original set of approximation functions is necessary, when adding or removing specific modes to the element basis.

A POSTERIORI ERROR ESTIMATION

For complex problems where no analytical solution is readily available, a local refinement of elements within a discretization is normally based on a posteriori error estimators. Computations and refinement of the infinite elements in this contribution are based on rather basic error estimators. An estimation of the error may be obtained by means of explicit residual methods. Kelly et al. [13] proposed an error estimator where the error is defined by the interior residual and the boundary residual. Regarding the enrichment of the infinite elements, only the boundary residuals on the envelope are considered for the corresponding computations in this contribution.

Another approach to estimate the error may be obtained from the fact that with increasing order of the infinite elements the quality of the numerical solution also increases. Hence, the difference between two numerical solutions with infinite elements of different order may serve as an error indicator (cf. [9] and the references therein for details). Advantages of this method are the rather general applicability and the rather easy implementation. An obvious drawback, however, is the fact that the original problem has to be solved for two different discretizations before an indication of the error is obtained.

NUMERICAL EXAMPLES

As mentioned earlier, the infinite elements generally try to resemble the analytic solution of the Helmholtz equation in an unbounded domain. Furthermore, infinite elements of radial order m can model the sound radiation behavior of a multipole of similar order exactly. In order to investigate the capabilities of a locally varying radial order of the infinite elements, the example investigated here is the sound radiation of a spherical structure of radius $R_S = 10$. The surrounding fluid is discretized with one layer of finite elements. The infinite elements are then attached at the envelope located at $R_\Gamma = 11$. Figure 2 (c) depicts a quarter of the discretization. Normal velocity boundary conditions corresponding to a specific multipole are prescribed on the inner boundary. For this example an analytic solution can be derived. Taking the diameter of the sphere as a reference length, the non-dimensional wave number is $kR_K \approx 2$. The computations were performed employing the FE-library libMesh [14].

The results given in Figure 2 (a) and (b) depict distributions of the estimated and the actual error on the envelope for the sound field generated by a quadrupole, where a residual based error estimator has been employed. Qualitatively, the estimated error matches the distribution of the actual error. Corresponding to the error indication, the radial order has been locally increased as depicted in Figure 2 (d).

The computed sound pressure amplitude in circumferential direction for infinite elements of order $m = 1$, $m = 2$ and for the discretization with varying order are shown in Figure 3.

In a second example, the difference of two numerical solutions obtained from simulations with different radial order is employed as an error indicator. First, the problem is solved for infinite elements of order $m = 0$ and $m = 1$. In subsequent steps, the order is increased, if the difference exceeds a certain criteria. For the sound field generated by a multipole of order 5, Figure 4 depicts the error and the resulting refinement up to order $m = 5$ on the envelope.

In order to monitor the numerical solution, a scaled L_2–error within the conventional finite elements has been computed. The error is generally defined by $\|p_{ex} - p_h\|_{L_2}/\|p_{ex}\|_{L_2}$,

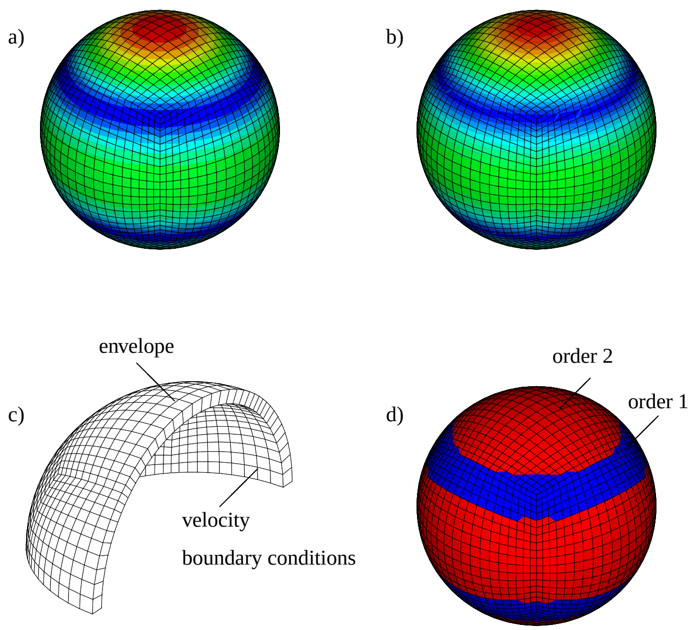

Figure 2: Estimated error (a) and actual error (b) distributions, FE model (c) and enrichment of the discretization(d).

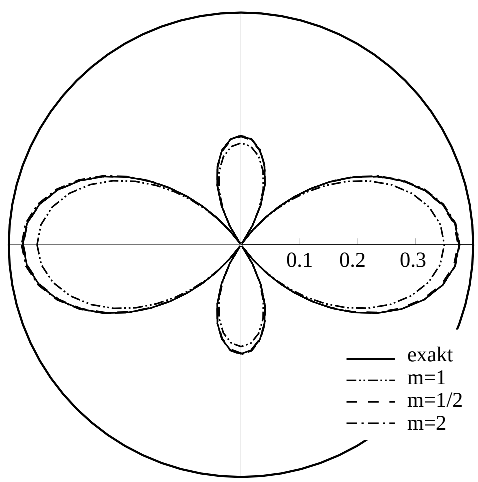

Figure 3: Sound pressure amplitude for discretizations with different radial order m and analytically exact solution.

where p_h is the finite element solution and p_{ex} is the analytical solution. Comparing the numerical solution obtained from the radial orders depicted in Figure 4 (with 9949 radial degrees of freedom) with the solution for infinite elements of order $m = 5$ (with 12010 radial degrees of freedom), the difference of the L_2–error occurred to be less than 1%.

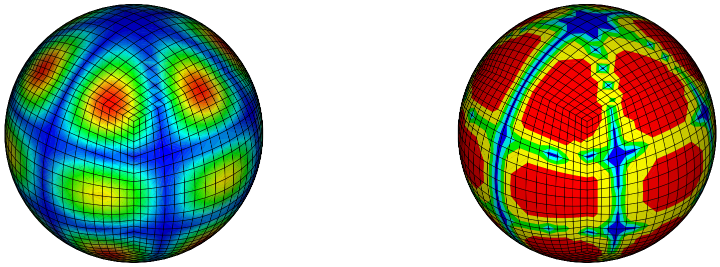

Figure 4: Error (left) and radial order of infinite element varying from 1 to 5 (right).

SUMMARY

The concept presented in this contribution enables a locally refinement of radial order of the infinite elements. In order to fully exploit the advantages of the adaptive refinement, further investigations regarding the a posteriori error estimation are necessary.

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