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INVERSION-FREE SOUND SOURCE RECONSTRUCTION

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Abstract
The ability to reconstruct a sound source using data measured in the nearfield of the source surface is clearly of great benefit in studying vibration and sound. Such reconstructions pose an interesting numerical problem because processing the measured data requires an inversion which is very often singular. The singular behavior arises from the loss of information from modes which decay rapidly away from the source and can be detected only in the nearfield of the source. By contrast, the process of constructing sound fields by propagating away from the source surface is relatively straightforward, and even provides a natural filter of the non-radiating modes. In this paper we present a new method of source reconstruction which uses propagation outward, with its natural filtering, together with a simple iteration scheme, to reconstruct a sound source. The source surface may or may not be a closed surface. Tests indicate that very accurate reconstructions are possible.

INTRODUCTION

For the calculation of the sound radiation from vibrating structures, the boundary element method (BEM) applied to the surface Helmholtz integral equation (SHIE) has become a common standard[1]. Research continues[2, 3] with the goal of improving the speed of calculation for cases involving a large number of elements ($N > 1000$); BEM calculations require the time intensive procedure of inverting a complex $N \times N$ matrix, which in some cases may be singular. Recently attention has been directed toward revisiting older methods employing eigenfunction expansions, with particular interest in spherical wavefunctions (SWF)[4, 5]. While the first such methods used the orthogonality of the eigenfunctions on a spherical surface, more recent methods employ fits to data on arbitrarily shaped surfaces[6, 7]. A difficulty with such fits to non-spherical surfaces, as will be discussed in this paper, is that the fits are limited to relatively low orders of the functions, with the result that outwardly propagated wave fields, and more particularly inversely propagated wavefields, are rather inaccurately

calculated. In this paper we present a new method, based on spherical wavefunctions, "iterative deepening" and "forward refinement"[8] using the surface Helmholtz integral equation, which has the following significant advantages: a) The method requires the inversion of only one $N \times M$ matrix, with $M\varphi N/10$, resulting in large time savings for $N > 1000$. b) The method works even at frequencies for which there are internal resonances (when the SHIE does not have a unique solution), and does not require the use of internal points as in the combined Helmholtz integral formulation (CHIEF) method[9]. For the singular cases it is usually not necessary to calculate extra functions as in the Burton-Miller method[10]. c) The method can produce accurate results (at the level of a few percent) for inverse as well as forward propagation calculations; even the inverse propagation requires only the single $N \times M$ matrix inversion. The key to this method lies in overcoming the limitation on the number of spherical wavefunctions which can be used to fit the data for non-spherical surfaces. The origin of the limitation may be understood as follows:

We consider experimental data for a sound field measured on a surface, e.g. the surface normal velocity measured on a vibrating surface, or the sound pressure measured on a "hologram" surface. The data is measured at discrete points, and some assumption must be made concerning the behavior of the field on the continuous surface between the measured points. The only reasonable assumption is that the field varies slowly between the measured points; for example, it should be the case that the actual field could be accurately represented by the measured data points and quadratic shape functions. One of the reasons for the great success of the BEM is that it takes advantage of the assumption that fields vary according to shape functions in evaluating integrals for the SHIE over the measured data. By contrast, it is in satisfying this assumption, for non-spherical surfaces, that spherical wavefunction fits are of limited value. For non-spherical surfaces, high order functions would be required simply to conform to the location of the data points (e.g. fitting relatively "sharp" corners) in addition to fitting the values at the data points. The result is that while many high order functions could well fit the data at the measured points, the field so represented between the data points would vary rapidly, not satisfying the assumption that the actual field varies slowly. An important point is that this problem cannot be solved by simply having a high density of data points; in this case the fit with the spherical wavefunctions may require a very large number of terms, each with very small coefficients, and such coefficients may be too difficult to determine in practice. Mathematically, representing a function with a complete set of eigenfunctions requires an infinite sum, and the sum may converge too slowly for numerical implementation with a truncated series. To avoid unlikely rapidly varying features resulting from high order terms in a fit with spherical wavefunctions, fits must be limited to low orders. Such low order fits provide a correspondingly inaccurate representation of the shape and value of the field on the surface, in error by 20% to 100%.

In our new method for solving forward and inverse radiation problems for non-spherical sources, fitting with spherical wavefunctions is a critical part of the solution, but the fit is not the final solution, as will be explained in this paper. Our research for non-spherical shapes, such as the "muffler" and "engine" shown in Figure 1, and for values of ka (wavenumber times characteristic source size) up to 25, shows that this method reproduces the results of a standard BEM calculation for forward propagation to better than three percent.

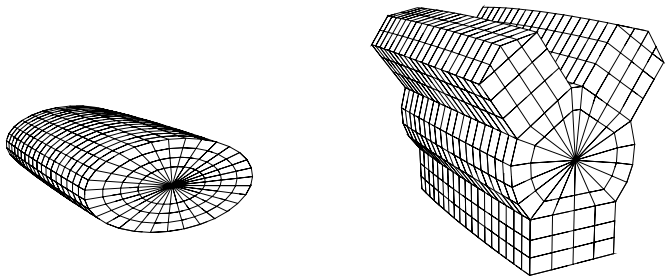

Figure 1: Non-spherical shapes, designated a "muffler" and an "engine", which have been studied with the new method for calculating sound fields. Starting with $N\varphi 1000$ points for a boundary condition, sound fields can be quickly calculated by inverting a single $N \times M$ matrix, with $M\varphi N/10$. There are no problems with singular matrices and no necessity for selecting interior "CHIEF" points.

SPHERICAL WAVEFUNCTIONS

Separating variables in spherical coordinates yields the spherical wave functions. For finite size sources, one has the boundary condition that solutions be outgoing waves at infinity. Thus for the spherical wavefunctions (assuming a time dependence $\exp(i\omega t)$) we use

$$\Phi_{lm}(r,\theta,\phi) = (j_l(r) - iy_l(r))\, e^{im\phi} Y_{lm}(\cos\theta) \tag{1}$$

where

$$Y_{lm}(\zeta) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\zeta) \tag{2}$$

The j_l and y_l are the spherical Bessel functions, and the P_l^m are the associated Legendre polynomials. The Y_{lm} are similar to the spherical harmonics, but with the ϕ dependence removed; the ϕ dependence is included explicitly in Eq. (1).

Any solution to a radiation problem may be written as a linear combination of these functions:

$$p(\vec{r}) = \sum_l \sum_m A_{lm} \Phi_{lm}(\vec{r}) \tag{3}$$

A necessary field is the normal component of the particle velocity at a surface given by points $\vec{r}_s$. We define a surface field which incorporates the acoustic impedance ρc as $v(\vec{r}_s) = \rho c \hat{n}(\vec{r}_s) \cdot \vec{v}(\vec{r}_s)$, where $\hat{n}(\vec{r}_s)$ is the unit normal at the surface point $\vec{r}_s$. In terms of the eigenfunctions Φ_{lm}, we have

$$v(\vec{r}_s) = \sum_l \sum_m A_{lm} \Phi'_{lm}(\vec{r}_s) \tag{4}$$

where $\Phi'_{lm}(\vec{r}_s) = (1/k)\, \hat{n}(\vec{r}_s) \cdot \vec{\nabla} \Phi_{lm}(\vec{r}_s)$.

LOW ORDER FITS WITH EIGENFUNCTIONS

For the forward propagation problem, we assume that the normal component of the particle velocity field is known at a finite set of discrete points $\vec{r}_i$ with $i = 1, 2, 3, ...N$. We simplify notation by writing $v(\vec{r}_i) = v_i$, and similarly for other functions of $\vec{r}_i$. We simplify notation by letting the one integer μ index the two subscripts l and m : $\mu = [l(\mu), m(\mu)]$. We truncate to a finite number of basis functions, so that $\mu = 1, 2, 3, ...M$. Finally, we use the convention that an index repeated in a term indicates a sum over that index. Now we can rewrite Eq. (4) as $v_i = \Phi_{i\mu}{}' A_\mu$. v_i is a complex vector of length N, A_μ is a complex vector of length M, and $\Phi'_{i\mu}$ is an $N \times M$ matrix. We shall later find that $M \varphi N/10$, so that the matrices are not square. The method of Singular Value Decomposition (SVD) allows one to find the inverse of such matrices. This is equivalent[11] to adjusting the coefficients A_μ to least-squares fit the boundary data v_i. Once the A_μ have been determined from the surface normal velocity v_i, then the surface pressure may be determined with $p(\vec{r}_i) = p_i = \Phi_{i\mu} A_\mu$. In holography, it is usually the surface pressure on a hologram surface which is measured. In this case the equation for p_i is inverted to find A_μ, from which v_i may be calculated.

What has been discussed above is not new; it is essentially the Rayleigh-Ritz method, discussed in basic texts on mathematical methods[4], implemented with contemporary computer algorithms. Our technique evolved from the method of Koopmann[12] and was at first referred to as the "equivalent source method."[7, 13] Other researchers had used similar methods before our implementation[5, 6]. Wu[14, 15] has extensively refined and extended the technique. What is new in our current method is the procedure of "forward refinement"[8] with "parametric relaxation". The application of "iterative deepening"[8] for solving boundary value problems is also new.

As discussed earlier, the discrete nature of data points on a non-spherical surface limits the order (and the number of basis functions M) which may be found by least-squares fitting the data. This limit causes the vectors v_i and p_i to be relatively inaccurate representations of the actual sound fields. The method for determining the optimal order for least-square fits to the surface data, as well as the method of "forward refinement" for converting v_i and p_i to accurate versions, employs the BEM matrices for the SHIE. These are discussed next.

THE BEM MATRICES

The surface Helmholtz integral equation (SHIE) for the sound pressure $p(\vec{r}_i)$ is[1]

$$p(\vec{r}_i) = \frac{1}{\Omega(\vec{r}_i)} \left(\iint \left(-\hat{n} \cdot \vec{\nabla}_s G(\vec{r}_i; \vec{r}_s) \right) p(\vec{r}_s)\, dS \right)$$

$$+ \frac{1}{\Omega(\vec{r}_i)} \left(\iint \left(-ikG(\vec{r}_i; \vec{r}_s) \right) v(\vec{r}_s)\, dS \right) \tag{5}$$

where $\Omega(\vec{r}) = \iint \hat{n} \cdot \vec{\nabla}_s G_0(\vec{r}; \vec{r}_s)\, dS$, $G(\vec{r}; \vec{r}_s) = \exp(-ik \mid \vec{r} - \vec{r}_s \mid) / 4\pi \mid \vec{r} - \vec{r}_s \mid$, $G_0(\vec{r}; \vec{r}_s) = 1/4\pi \mid \vec{r} - \vec{r}_s \mid$ and $\hat{n} \cdot \vec{\nabla}_s$ ia a normal derivative at the point $\vec{r}_s$.

Taking the derivative of this equation to obtain a similar integral equation for the surface normal velocity $v(\vec{r}_i)$ results in singular integrands. These singularities may be removed by subtracting kernels with similar singularities and by having factors which vanish at the singular points[1]. In the boundary element method, the surface is divided into elements, with a number of nodes on their peripheries. The integrals over the elements are accomplished using shape functions and/or constant elements (with the field over the entire element taking on the value of the field at the center of the element). When this is done the SHIE and its derivative become matrix equations:

$$p_i = A_{ij}p_j + B_{ij}v_j \tag{6}$$

$$v_i = C_{ij}p_j + D_{ij}v_j \tag{7}$$

The matrices A, B, C, D are integrals of the shape functions and/or various kernels (G, etc.) over individual elements, and the vectors p and v are the sound fields evaluated at the element nodes or centers.

FORWARD REFINEMENT, ETC.

To describe forward refinement, as well as the procedures of iterative deepening and parametric relaxation, the case of forward propagation, where v is known and p is to be determined, will be used as an example. A simple version of forward refinement with iterative deepening will be discussed first, and then this procedure itself will be refined with "parametric relaxation". To clarify the procedure, we re-write Eq. (6) as

$$p_i^{out} = A_{ij}p_j^{in} + E_i \tag{8}$$

where $E_i = B_{ij}v_j$. The basic procedure is as follows: a) Start with a basis set of SWF Φ_{lm} using $l = 0$ through $l_{\max}$, where $l_{\max}$ is a low order (e.g. 3). b) Use least-squares fitting of the known v_i with the low order basis set to find the relatively inaccurate fields v^{swf} and p^{swf}. c) Set $p^{in} = p^{swf}$. d) Using p^{in} and Eq. (8), calculate p^{out}. Also calculate $S = \Sigma_i \left(p_i^{in} - p_i^{out}\right)^2$. e) Set $p^{in} = p^{out}$, and iterate by returning to step d). Terminate the iteration if S starts to increase. f) Save the best value of S for this $l_{\max}$ as S_l, and increase $l_{\max}$. However, before returning to step b) for an outer iteration loop, remove functions from the basis set which had relatively low values of the coefficients in the fit

for v^{swf}. This reduces the value of M and decreases the time required to invert the $N \times M$ matrix in the least-squares fit. This outer iteration loop, in which $l_{\max}$ is increased, continues until S_l starts to increase.

The term "iterative deepening" refers to the pruning of the basis set and the reduction of M for the next value of $l_{\max}$ in step f); the term is taken from a procedure in computer chess, where good moves found in a shallow search of a tree of possible moves are used to optimize a search to a greater depth[16]. From the minimum of S_l, the optimum value of $l_{\max}$ is found, usually resulting in $M\varphi N/10$.

Step d) is referred to as "forward refinement" because the calculation is done in a "forward direction" with the original matrices, rather than their inverses. This procedure so far is

the same as solving the matrix equation Eq. (6) by iteration[2], using p^{swf} as a seed; however, the procedure becomes significantly different with the addition of parametric relaxation, discussed next.

If the iteration in step d) were undertaken as described, it is likely that the iteration would not converge to the desired solution. Parametric relaxation is used to guarantee the desired convergence. First of all, general convergence could be improved by not using $p^{in} = p^{out}$ in step e), but instead using $p^{in} = \left(p^{out} + p^{prev}\right)/2$, where p^{prev} was the previous version of p^{in} which produced the current version of p^{out}. Further convergence to the desired solution may be obtained by using
$p^{in} = \left[p^{out} + (1-\beta)\, p^{prev} + \beta p^{swf}\right]/2$ with the parameter β between 0 and 1. Even though p^{swf} is relatively inaccurate, it still gives the iteration an excellent idea of where it should be heading. In any case, convergence to the desired solution is further guaranteed by replacing $E_i = B_{ij} v_j$ with $E_i{}' = B_{ij}\left[(1-\alpha)\, v_j + \alpha v_j^{swf}\right]$, with the parameter α between 0 and 1. Consider the case when $\alpha = \beta = 1$; this corresponds to starting with p^{swf} and v^{swf} in Eq. (6). Since p^{swf} and v^{swf} are linear combinations of spherical wavefunctions, they exactly satisfy the SHIE, and we have $S_l = S = 0$ exactly, clearly the global minimum. If α and/or β are changed slightly, then the global minimum will shift slightly, but the previous solution will not be far off, so that the iteration procedure should bring it to the new global minimum of S. Thus by slowly relaxing α and β to zero, the iterative solution will track the global minimum of S, and converge to the desired solution. This slow variation of α and β from 1 to 0 is the parametric relaxation procedure. In the current implementation of the computer program, β is set at 1 during the iterative deepening process (finding the optimum $l_{\max}$) and α is relaxed from 1 to 0 in steps of 0.2. Once the optimum $l_{\max}$ is found, β is relaxed from 0.9 to 0 in steps of 0.1, and at each value of β, α is relaxed as before. It is important to note that in order to find the accurate version of the unknown surface field, it is crucial to know p^{swf} and v^{swf}; while they may be relatively inaccurate versions of the actual wave fields, they are still approximations of the actual wave fields which are exact solutions of the SHIE, and thus serve to initially locate the global minimum of S.

For holographic reconstruction, where p is measured on a hologram surface, the SWF and forward refinement are used as above to find an accurate version of the surface normal velocity v; the only difference is that the roles of Eq. (6) and Eq. (7) are reversed.

INVERSE PROPAGATION

In holographic sound source reconstruction, the sound pressure p_h measured on a hologram surface $\vec{r}_h$, with $p_h = p\left(\vec{r}_h\right)$, must be used to determine the normal velocity on the surface of a source, $v_s = v\left(\vec{r}_s\right)$. Here it will be assumed that the hologram measurements are taken in the nearfield, very close to the source; in our current research we assume measurements are taken on a congruent surface at a distance $\mid \vec{r}_h - \vec{r}_s \mid \varphi 0.1 \mid \vec{r}_s \mid$.

The first step is to use the normal derivative surface Helmholz integral equation (ND-SHIE, Eq. (7)) and p_h to determine the normal velocity v_h at the hologram surface. This can be done with a boundary element method; while the Dirichlet problem is more difficult than

the Neumann one (SHIE) for the BEM, one at least has an analytical equation which can be solved with the BEM. For the remainder of the inverse problem, one does not have an analytical equation to address.

The next step is to fit p_h and v_h with spherical wavefunctions. With the coefficients of this fit one can evaluate SWF fields on the hologram surface and on the source surface: $p_h^{swf},\ v_h^{swf}$ and $p_s^{swf},\ v_s^{swf}$. These fields are then used to seed an iteration proceedure as follows:

a) The hologram fields at individual points, p_{hi} and v_{hi}, are transferred back to corresponding points on the source surface with

$$p_{si}{}' = \left(p_{si}^{swf}/p_{hi}^{swf}\right) p_{hi} \quad \text{and} \quad v_{si}{}' = \left(v_{si}^{swf}/v_{hi}^{swf}\right) v_{hi} \tag{9}$$

This simple proceedure alone can result in a 10% - 30% reconstruction of the source. At a few points the denominators in the equations may be zero, and at these points one uses the equations

$$p_{si}{}' = \left(p_{si}^{swf}/v_{hi}^{swf}\right) v_{hi} \quad \text{or} \quad v_{si}{}' = \left(v_{si}^{swf}/p_{hi}^{swf}\right) p_{hi} \tag{10}$$

In the rare case any of these ratios are also singular, one simply uses the average of non-singular ratios at nearby points.

b) The fields $p_{si}{}'$ and $v_{si}{}'$ may be refined with Eqs. (6) and (7), to ensure that they obey the surface Helmholtz integral equation.

c) The next step is to use the Helmholtz integral equation on the source surface to propagate $p_{si}{}'$ and $v_{si}{}'$ out to the hologram surface, to obtain the fields $p_{hi}{}'$ and $v_{hi}{}'$.

d) One now loops back to step a), but with the SWF fields replaced with the new fields $p_{hi}{}',\ v_{hi}{}'$ and $p_{si}{}',\ v_{si}{}'$, and the iteration continues.

The iteration is monitored with $\mid p_{hi} - p_{hi}{}' \mid^2$, and is terminated when this starts to increase. It may be possible to reduce the growth of unwanted features by smoothing the iteration fields by adding some fraction of the SWF fields (as with the parameter β in the iterative solution of the SHIE). However, we have not yet implemented this feature in our current computer code.

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