FURTHER RESULTS FOR STATE SPACE FLUID-STRUCTURE ANALYSIS USING COUPLED BOUNDARY AND FINITE ELEMENT ANALYSES

John B. Fahnline

Applied Research Laboratory, Pennsylvania State University
University Park, PA 16803
jbf103@only.arl.psu.edu

Abstract
In the field of structural-acoustics, there is considerable interest in finding an efficient method for computing fluid-coupled mode shapes, resonance frequencies, and loss factors. The computations are difficult because the acoustic field created by a vibrating structure is generally not a simple function of frequency, making the associated eigenvalue problem nonlinear. A state-space formulation has been used previously to solve structural-acoustic eigenvalue problems for combined finite element / boundary element analyses. In this paper, the formulation is adapted for the software package ARPACK by generalizing the previous results for arbitrary polynomial order and developing simple formulas for the matrix inverse and matrix-vector multiplication required within ARPACK. Using an example problem of a circular cylinder, it is shown that higher order polynomial expansions can actually increase the overall computational efficiency because they allow a wider frequency range to be represented during a single iteration, resulting in fewer subdivisions of the overall frequency range.

INTRODUCTION

The acoustic field created by a vibrating structure is generally not a simple function of frequency, so that the associated eigenvalue problem is nonlinear. Giordano and Koopmann [1] derived a solution to the eigenvalue problem by representing the surface pressure coupling using boundary elements and assuming that each term in the resulting pressure-to-velocity transfer function matrix could be approximated over frequency bands using cubic polynomials. By adding the acoustic pressure field into the equations of motion and converting the eigenvalue problem to state-space, they showed that a solution for the fluid-coupled modes could be computed. Subsequently, Cunefare and De Rosa [2] improved the method slightly by simply writing the boundary element matrices in terms of pressure-to-displacement transfer functions, reducing the state-space system to three times the number of displacement variables rather than four.

In isolation, these papers seem to indicate that the state-space analysis is effective, but computationally-inefficient, since the matrices to be analyzed are three or four times as large as the number of displacement degrees-of-freedom (either in physical or modal coordinates). However, in the extended analysis in his thesis, Giordano [3] shows that the eigenvalue problem can be solved efficiently using a shifted inverse iteration algorithm, requiring only a single matrix inverse of the size of the number of displacement degrees-of-freedom. In this form, the matrix inverse dominates the solution times and the computational efficiency is not heavily dependent on the size of the state-space system of equations. Giordano goes on to suggest that the basic state-space methodology could be extended to higher order polynomials, and that frequency normalization might be required to prevent the coefficients from exceeding machine precision.

In this paper, the numerical method developed by Giordano will be re-examined, with the goal of extending the analysis and simplifying its application within ARPACK [4]. This numerical package is ideal for the problems under consideration because the input matrices can be unsymmetric and complex-valued, and it is designed to compute a limited number of eigenvalues near a specified value. The main contributions of the present work will be to generalize the results in Giordano's thesis for arbitrary polynomial order and to develop simple formulas for the matrix inverse and matrix-vector multiplication required within ARPACK. Further, it will be shown that higher order polynomial expansions can actually increase the computational efficiency of the algorithm because they allow a wider frequency range to be represented during a single iteration, resulting in fewer subdivisions of the overall frequency range.

INVERTING THE STATE-SPACE MATRIX

The analysis begins from the equations of motion for a vibrating structure including surface pressures due to fluid coupling, which can be written as

$$\left[\mathbf{K} - \omega^2 \mathbf{M} + \mathbf{A}(\omega) \right] \mathbf{d} = \mathbf{f} \ , \tag{1}$$

where **K** and **M** are the structural stiffness and mass matrices, respectively, **A** represents the surface pressure forces due to sound radiation from a vibrating structure, **d** is the displacement vector, and **f** is the input force. The **K** and **M** matrices are typically generated using a finite element analysis and the **A** matrix can be generated using either finite or boundary elements. Thorough discussions of the methods for generating the matrices in these applications have been given by Giordano [1, 3], Cunefare [2] and Fahnline [5]. In a practical implementation, the matrices will be written in modal, rather than displacement, coordinates by expanding the displacement in terms of the *in vacuo* structural modes. Because the fluid-coupling matrix **A** is an arbitrary (but continuous and smoothly-varying) function of frequency, the associated eigenvalue problem is nonlinear.

As was demonstrated by Giordano in his Ph.D. thesis, the eigenvalue problem can be linearized and converted to state-space by expanding each term of the coefficient matrix as a polynomial function in ω. To minimize space requirements, quadratic polynomials will be used in the following derivation and the results will be generalized for abitrary order subsequently. Writing out the matrix **A** as a polynomial in ω gives

$$\mathbf{A}(\omega) = \mathbf{A}_0 + \mathbf{A}_1 \omega + \mathbf{A}_2 \omega^2 \ , \tag{2}$$

where **K** and **M** have been absorbed into $\mathbf{A}_0$ and $\mathbf{A}_2$, respectively. Post-multiplying by the displacement vector and noting that a time-derivative is equivalent to multiplication by - i ω if **d** is time-harmonic, Equation (2) can be rewritten as

$$\mathbf{A}(\omega)\mathbf{d} = \mathbf{A}_0\mathbf{d} + \frac{\mathbf{A}_1}{-\mathrm{i}}(-\mathrm{i}\,\omega\,\mathbf{d}) - \mathbf{A}_2\left(-\omega^2\mathbf{d}\right) = \mathbf{A}_0\mathbf{d} + \mathrm{i}\,\mathbf{A}_1\,\dot{\mathbf{d}} - \mathbf{A}_2\,\ddot{\mathbf{d}}. \tag{3}$$

Pre-multiplying by $-\mathbf{A}_2^{-1}$ and taking $\mathbf{f} = 0$, Equation (1) reduces to

$$\ddot{\mathbf{d}} - \mathrm{i}\,\mathbf{A}_2^{-1}\mathbf{A}_1\,\dot{\mathbf{d}} - \mathbf{A}_2^{-1}\mathbf{A}_0\mathbf{d} = 0 \ . \tag{4}$$

Writing the matrix system as a standard eigenvalue problem in state-space gives

$$\begin{bmatrix} -\mathrm{i}\,\mathbf{A}_2^{-1}\mathbf{A}_1 - \mathrm{i}\omega\,\mathbf{I} & -\mathbf{A}_2^{-1}\mathbf{A}_0 \\ -\mathbf{I} & -\mathrm{i}\omega\,\mathbf{I} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{d}} \\ \mathbf{d} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \ . \tag{5}$$

To solve this eigenvalue problem using ARPACK, the matrix in square backets has to be inverted and multiplied by a vector during each iteration. Because the solution of a densely-populated matrix system is proportional to its size cubed, it is very inefficient to directly invert the state-space matrix, especially if higher order polynomials are used in the expansion. However, since the solution has been assumed to be time-harmonic, the displacement derivatives are not independent variables, and it is possible to write the derive a semi-analytical solution for the inverse by partitioning the matrix into submatrices.

To start, the inverse can be written in terms of its submatrices as

$$\begin{bmatrix} -\mathrm{i}\,\mathbf{A}_2^{-1}\mathbf{A}_1 - \mathrm{i}\omega\,\mathbf{I} & -\mathbf{A}_2^{-1}\mathbf{A}_0 \\ -\mathbf{I} & -\mathrm{i}\omega\,\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{I} \end{bmatrix} , \tag{6}$$

where the submatrices of **A** are known and those of **B** are to be determined. Multiplying the two matrices in Equation (6), the second row of the result yields

$$-\mathbf{B}_{11} - \mathrm{i}\omega\,\mathbf{B}_{21} = 0 \quad \text{and} \quad -\mathbf{B}_{12} - \mathrm{i}\omega\,\mathbf{B}_{22} = \mathbf{I} \ . \tag{7}$$

Solving for $\mathbf{B}_{11}$ and $\mathbf{B}_{12}$ and substituting into Equation (6) gives

$$\begin{bmatrix} -\mathrm{i}\,\mathbf{A}_2^{-1}\mathbf{A}_1 - \mathrm{i}\omega\,\mathbf{I} & -\mathbf{A}_2^{-1}\mathbf{A}_0 \\ -\mathbf{I} & -\mathrm{i}\omega\,\mathbf{I} \end{bmatrix} \begin{bmatrix} -\mathrm{i}\omega\,\mathbf{B}_{21} & -\mathbf{I} - \mathrm{i}\omega\,\mathbf{B}_{22} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{I} \end{bmatrix} . \tag{8}$$

Because each column of the inverse depends only on a single matrix, it is possible to directly solve for the submatrices. This is true regardless of the number of terms included in the polynomial expansion, and thus a similar technique can always be used to compute the inverse. Solving for $\mathbf{B}_{21}$ and $\mathbf{B}_{22}$ gives

$$\mathbf{B}_{21} = -\,\mathbf{A}(\omega)^{-1}\,\mathbf{A}_2 \quad \text{and} \quad \mathbf{B}_{22} = \mathrm{i}\,\mathbf{A}(\omega)^{-1}\left(\mathbf{A}_1 + \omega\,\mathbf{A}_2\right) \ . \tag{9}$$

The submatrices of the matrix inverse can now be computed without having to invert the full state-space matrix, making this method computationally-efficient.

To be able to use ARPACK to compute solutions to eigenvalue problems, the user is supplied an input vector and he multiplies it by the matrix inverse and returns the result. For the present problem, it is not necessary to actually construct the matrix inverse and perform the multiplication. Using a quadratic polynomial expansion, the inverse can be written in more condensed form as

$$\begin{bmatrix} -\mathrm{i}\omega\,\mathbf{B}_{21} & -\mathbf{I} - \mathrm{i}\omega\,\mathbf{B}_{22} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} = \begin{Bmatrix} -\mathrm{i}\omega\,\mathbf{I} \\ \mathbf{I} \end{Bmatrix} \begin{Bmatrix} \mathbf{B}_{21} & \mathbf{B}_{22} \end{Bmatrix} + \begin{bmatrix} 0 & -\mathbf{I} \\ 0 & 0 \end{bmatrix} . \tag{10}$$

Post-multiplying by the input displacement vector then yields

$$\begin{bmatrix} -\mathrm{i}\omega\,\mathbf{B}_{21} & -\mathbf{I} - \mathrm{i}\omega\,\mathbf{B}_{22} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{d}} \\ \mathbf{d} \end{Bmatrix} = \begin{Bmatrix} -\mathrm{i}\omega\,\mathbf{I} \\ \mathbf{I} \end{Bmatrix} \left\{ \mathbf{B}_{21}\,\dot{\mathbf{d}} + \mathbf{B}_{22}\,\mathbf{d} \right\} + \begin{Bmatrix} -\mathbf{d} \\ 0 \end{Bmatrix} . \tag{11}$$

In this form, several matrix-vector multiplications are performed, but the matrices are only the size of the number of displacement degrees of freedom.

GENERALIZED FORMULAS

The preceeding analysis can be performed for several polynomial orders and generalized by induction. Taking N to be the exponent for the highest order polynomial for ω, the result for the matrix inverse is given as

$$\mathbf{B}_{Nm} = i^{(N-m-1)}\ \mathbf{A}^{-1} \sum_{\mu=N-m-1}^{N} \omega^{(\mu-N+m+1)}\ \mathbf{A}_{\mu}\ ,\ m = 1, \ldots, N\ . \quad (12)$$

Similarly, the result for the matrix-vector multiplication can be generalized as

$$\mathbf{y}_m = (-i\omega)^{(N-m)} \sum_{\mu=1}^{N} \mathbf{B}_{N\mu}\,\mathbf{d}_{\mu} - \sum_{\nu=m+1}^{N} (-i\omega)^{(\nu-m-1)}\,\mathbf{d}_{\nu}\ . \quad (13)$$

In Equation (13), the $\mathbf{d}_{\mu}$ are the components of the displacement vector in state-space, and thus $\mathbf{d}_N = \dot{\mathbf{d}}$, $\mathbf{d}_{N-1} = \ddot{\mathbf{d}}$, etc.

In his thesis, Giordano also discusses the possibility of normalizing the frequency scale for higher order polynomial expansions. Numerical experiments have verified that frequency normalization is necessary for fourth order polynomials and higher. Normalizing the frequency by the upper limit of the range for the expansion as $\Omega = \omega / \omega_{max}$, the polynomial expansion can then be written as

$$\mathbf{A}(\Omega) = \mathbf{A}_0 + (\omega_{max}\ \mathbf{A}_1)\ \Omega + \left(\omega_{max}^2\ \mathbf{A}_2\right) \Omega^2 + \ldots\ . \quad (14)$$

Thus, the only change that is required is to multiply the input matrices by the appropriate power of ω_{max}. After solving for the eigenvalues in terms of Ω, those for the original problem can be determined as $\omega = \omega_{max}\ \Omega$. These formulas have been tested for different polynomial orders and they yield nearly identical results for the resonance frequencies, mode shapes, and damping loss factors.

ELIMINATING NUMERICAL ARTIFACTS

In general, it is not possible to represent the acoustic matrix as a simple polynomial function over the entire frequency range of interest, and it must be subdivided. Thus, a separate eigenvalue problem must be solved for each smaller frequency range, and the numerical efficiency is enhanced by maximizing the range for each of the subdivisions. Because polynomial interpolation is very fast and yields a simple accuracy estimate as discussed by Press, et al. [6], it is possible to test several frequency subdivisions and optimize the range to yield a specified accuracy for a given polynomial order.

The numerical efficiency can also be increased by choosing a reasonable lower bound on the frequency range to search for eigenvalues. Because the eigenvalue problem is formulated using *in vacuo* modes as basis functions, the resistive component of the acoustic matrices can have higher order frequency dependence at low frequencies. For example, the power output of a quadrupole source is

proportional to ω^4 at low frequencies, whereas that of a monopole source is proportional to ω^2. Thus, a lower order polynomial would not be able to properly represent the sound radiation from a quadrupole at low frequencies. This problem can be alleviated somewhat by choosing a reasonable lower bound on the frequency range, preventing the program from having to solve numerous eigenvalue problems within a frequency range where none exist.

There is some discussion in both Giordano's thesis and the papers by Giordano and Koopmann, and Cunefare about how to discriminate between actual modes and numerical artifacts. One method for making the identification process very reliable is to solve the eigenvalue problem several times using different polynomial orders. The resonance frequencies and damping levels for the actual modes are consistently repeated, while the numerical artifacts are not. In the present implementation, the eigenvalue problem is solved three times in each subdivision of the frequency range, using one less and one more polynomial terms than the number specified. For example, if the polynomial order is specified as four, the problem is solved using three, four, and five interpolation frequencies, with the interpolation accuracy determined using the lowest order polynomial.

EXAMPLE OF A CIRCULAR CYLINDER

As an example problem to test the accuracy of the formulas and to assess the efficiency of the computations, fluid-coupled resonance frequencies and modes shapes were calculated for a thin circular shell and compared to experimental measurements. In the experimental configuration, the shell is mounted to two solid metal endcaps, with rubber o-rings providing a water-tight seal between the interior and exterior volumes. A heavy steel bar runs along the centerline of the cylinder and through the endcaps to keep the cylinder in proper alignment and to support its weight. The shell's radius, length, and thickness are 6.38 inches, 46.75 iinches and 0.2047 inches, respectively, and it is made of steel. Numerical simulations and experimental measurements were performed for conditions (1) in air, (2) with air inside the shell and water outside, and (3) with water both inside and outside. Since the boundary conditions for the o-ring connections proved difficult to simulate numerically, the structural model was tuned to give the measured in-air resonance frequencies of the shell. The finite and boundary element models are shown on the left and right sides of Figure 1, respectively.

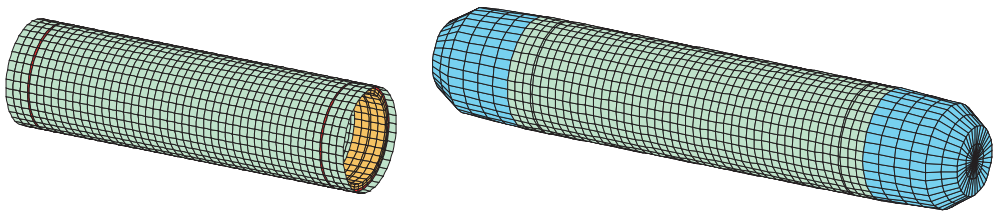

Figure 1. Finite element (left) and boundary element (right) models of the cylinder.

In the figure, the elements colored red in the finite element mesh are assigned the properties of rubber and are meant to simulate o-rings. The encaps are assumed to be motionless and are not included in the finite element model. In the boundary element model of the interior acoustic field, the cavity does not extend into the endcaps, shown in blue on the right side of Figure 1. One hundred and seventeen *in vacuo* modes from 0-3200 Hz are included as basis functions.

To demontrate the accuracy of the numerical computations, Table 1 lists experimental measurements and numerical simulations for the structural resonance frequencies of the shell, where N and M refer to the number of full waves around the circumference and half waves along its length, respectively.

Table 1. Measured and computed resonance frequencies for the cylinder.

			Air Inside, Water Out		Water Inside and Out	
		Air, Exp.	Exp.	Num.	Exp.	Num.
N	M	F_μ (Hz)	F_μ (Hz)	F_μ (Hz)	F_μ (Hz)	F_μ (Hz)
2	1	276	128	122	97	90
	2	530?	264	265	220	197
3	1	456	225	226	176	169
	2	660	340	335	261	251
4	1	827	450	453	350	345
	2	912	502	507	393	386
5	1	1321	771	791	608	611
	2	1375	806	831	639	642
6	1	1927	1184	1228	943	977
	2	1974	1218	1265	977	1008

Overall, the numerical simulations match the experimental measurements very well, especially considering the large shifts in the resonance frequencies when the shell is immersed in water. For the case with water both inside and out, acoustic resonances occur in the fluid within the cylinder, resulting in peaks in the matrix **A**. Near these frequencies, the polynomial expansion is limited to very small frequency ranges, so that numerous eigenvalue problems must be solved. Also, spurious eigenvalues are found near the acoustic resonances even when only the consistent roots are retained for several polynomial orders. Knowing the frequencies for the acoustic resonances, it is possible to eliminate the spurious roots, leaving the results shown in the table.

For the timing studies, the same computer was used for all the calulations, and the system clock was monitored during the program execution. As one might expect from the discussion in the previous paragraph, the analysis becomes inefficient near acoustic resonances because only a small frequency range can be represented accurately using low order polynomials. The results of the timing studies are listed in Table 2.

Table 2. Results of the timing studies.

	# Eigenvalue Problems		Computation Time (sec)	
# Interp. Freq.	**Water Outside**	**Inside & Out**	**Water Outside**	**Inside & Out**
4 (3, 4, 5)	3 x 57	3 x 321	832	4685
5 (4, 5, 6)	3 x 14	3 x 124	290	2569
6 (5, 6, 7)	3 x 7	3 x 80	265	3029
7 (6, 7, 8)	3 x 4	3 x 72	250	4500

The results show that the computation time is more controlled by the number of eigenvalue problems solved rather than the size of the state-space matrices. Higher order interpolation is beneficial when more of the frequency range can be represented in each eigenvalue analysis. The overall conclusion is that the state-space solution of the eigenvalue problem is computationally efficient as long as the variations in **A** can be represented using low order polynomials. Otherwise, it may be more efficient to model the interior acoustic field using finite elements.

CONCLUSIONS

In this paper, the state space algorithm for computing eigenvalues and eigenvectors has been generalized for arbitraray polynomial order. Formulas were derived for the inverse of the state space matrix and its multiplication by an input vector. This allows the eigenvalue problem to be solved efficiently using ARPACK. An example problem was used to show that the computational speed of the eigenvalue solution can actually be increased using higher order polynomials because a wider frequency range may possibly be represented during a single analysis.

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