THE SERIES METHOD FOR WAVE PROPAGATION IN HIPPOPLASTIC MEDUIM

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Abstract
A theoretical study of plane waves in hypoplastic medium is presented using a ray method of geometrical seismology. Dynamical equations for relative small disturbance about an initial state are derived from general equations of hypoplasticity. We consider the hypoplastic equations with intergranular strain and assume that the initial state and the density of medium are inhomogeneous. In this case using of the ray method is the most efficient. The solutions for the components of rate vector or stress are found in the form of ray series. The coefficients of these series satisfy on order differential equations of the transfer which are solved analytical in case when void ratio is less then unit.
The velocities of quasi-longitudinal and quasi-transverse waves in a hypoplastic medium with intergranular strain depend on a kind of the loading and the unloading and they are usually less than the velocities of longitudinal and transverse waves in elastic skeleton.

INTRODUCTION

Wave propagation equations in a hipoplastic medium allows to describes some effects which are observed in during earthquakes [1]. Constitutive equations with granular strain are strong nonlinear thus an application of small parameter method appear natural. In paper [2] authors applied the method of small parameter for solving of problems of dynamical hipoplastisity.

As the capacity of small parameter in [3] is taken the expression $\frac{\left\| T - T^{(0)} \right\|}{\left\| trT^{(0)} \right\|} = \varepsilon << 1$, where T, $T^{(0)}$ are stress tensor and initial stress tensor. The authors obtained the solutions of supplied problems at first approximation.

Theory of wave propagation in hipoplastic medium describes some stages in this process [4,5].

We consider a finding of next approximation.

If disturbance amplitudes of stresses are weak relative to initial stress state, so waves propagate as elastic waves in porous body with water. For homogeneous initial stress and constant density these wave propagate with constant effective velocities [6].

However already in this approximation we can consider plastic waves in case when grains of skeleton are under relative displacements with friction. Plastic wave propagate with rates, which are found in [6]. The model of a hipoplastic medium with intergranular strain gives a possibility to calculate the speeds of plastic waves [6].

Mathematical methods, which authors applied in [6], don't allow to consider the wave propagation in a hipoplastic medium for inhomogeneous initial stress state and variable density.

In that article we consider the ray method which is the greatest in geometrical seismology [5].

1.1 Mathematical formulation of wave propagation in hipoplastic medium.

Closed system of equations which describes a process of wave propagation in hipoplastic medium has the form in general case

$$\operatorname{div} T_{\text{total}} + \rho g = \rho \frac{\partial \overline{V}}{\partial t} \tag{1.1}$$

$$\frac{\partial \rho}{\partial t} + \nabla\left(\rho \overline{V}\right) = 0 \tag{1.2}$$

$$\dot{T} = M(T, e, \delta) : D - T\overline{W} + \overline{W}T \tag{1.3}$$

$$\dot{\delta} = F(\delta) : D - \delta\overline{W} + \overline{W}\delta \tag{1.4}$$

$$\dot{e} = (1+e)trD \tag{1.5}$$

The system (1.1) – (1.5) contains 17 equations for 17 field values: $\overline{V}$ is a velocity, *T* is a stress tensor, δ is a intergranular strain tensor, ρ is a density of medium, *e* is void ratio. For the system (1.1) – (1.5) are written initial and boundary condition for each concrete problem.

The system (1.1) – (1.5) is strong nonlinear therefore we consider some assumption which allow to simplify one. Assume analogous [6] that stress disturbances $\left\| T - T^{(0)} \right\|$ are less than initial stresses $T^{(0)}$

$$\frac{\left\| T - T^{(0)} \right\|}{\left\| T^{(0)} \right\|} \le \varepsilon << 1\,. \tag{1.6}$$

However, in contrast to [3], we don't assume that the initial stresses $T^{(0)}$ are constant.

Consider the case when $T = T^{(0)}(\bar{x})$, $\rho = \rho(\bar{x})$ are functions of spatial coordinate $\bar{x}$. Also we assume analogous [6] that initial stress state is hydrostatic state, then tensors $\hat{T} = T \cdot \|T\|^{-1}$, $\hat{T}^* = T \cdot \|T\|^{-1} - \frac{1}{3}J$ have the following form

$$\hat{T}_{ij} = \frac{1}{3}\delta_{ij}\ , \qquad \hat{T}^*_{ij} = 0 \tag{1.7}$$

From (1.7) it follows that $\hat{T}^*$ is the small parameter in nonlinear system (1.1) – (1.5). Applied the method of small parameter (the method of consequent approximation) we can find consequently the systems of equations for the approaches of field values.

At first consider equations (1.3), (1.4). Write these equations for condition (1.7) in the form [6]:

$$\dot{T} = \alpha_1 L : D + \alpha_2 L : \hat{\delta}\hat{\delta} : D + \alpha_3 N\hat{\delta} : D \tag{1.8}$$

$$\dot{\delta} = D - \alpha_4 \hat{\delta}\hat{\delta} : D \quad \text{, для } \hat{\delta} : D > 0 \tag{1.9}$$

$$\dot{T} = \alpha_5 L : D + \alpha_6 L : \hat{\delta}\hat{\delta} : D \tag{1.10}$$

$$\dot{\delta} = D \quad , \qquad \text{для } \hat{\delta} : D \le 0 \tag{1.11}$$

$$\alpha_1 = \rho_1^x m_T + (1-\rho^x) m_R\,, \quad \alpha_2 = \rho_1^x (1 - m_T), \quad \alpha_3 = \rho_1^x \ ,$$

$$\alpha_4 = \rho_1^{\beta_r}\ , \quad \alpha_5 = \alpha_1\ , \quad \alpha_6 = \rho_1^x (m_R - m_T),$$

$$L_{ijkl} = \lambda^* \delta_{ij}\delta_{kl} + 2\mu^* \delta_{ik}\delta_{jl}\ , \quad N_{ij} = \delta_{ij} X\,,$$

$$\lambda^* = \frac{f_b f_e a^2}{27}\ , \ \mu^* = \frac{f_b f_e}{6}\ , \ X = \frac{f_d f_b f_e}{27}.$$

$$f_b f_e \stackrel{def}{=} \frac{h_s}{h}\frac{1+e_i}{e}\left(-\frac{trT^{(0)}}{h_s}\right)^{1-n}\left[3 + a^2 - a\sqrt{3}\left(\frac{e_{io}-e_{do}}{e_{co}-e_{do}}\right)^{\alpha}\right]^{-1},$$

$$f_d \stackrel{def}{=} \left(\frac{e-e_d}{e_c-e_d}\right),\ a \stackrel{def}{=} \frac{\sqrt{3}(3-\sin\varphi_c)}{2\sqrt{2}\sin\varphi_c},\ \rho_1 = \frac{\|\delta\|}{R}.$$

The parameter φ_c corresponds to the critical friction angle [6].

1.2 The formulation of the problem.

Consider the problem of wave propagation in the layer of the hipoplastic medium $x_{10} \le x_1 \le x_{10} + L$ (Fig. 1). The thickness of the layer is L.

Practically the material is laminated, macroscopic effects of this lamination can be described by effectively model of medium. Let's consider cohesion of the layer with half-space rigid.

At the moment $t = t_0$ on the plane $x_1 = x_{10}$ we set the initial conditions.

$$V_1(x_{10}, t_0) = 0, \quad V_2(x_{10}, t_0) = V_2^0, \quad V_3(x_{10}, t_0) = 0\,. \tag{1.12}$$

$$\begin{gathered} T_{11}(x_{10}, t_0) = T_{22}(x_{10}, t_0) = T_{33}(x_{10}, t_0) = T_{13}(x_{10}, t_0) = T_{23}(x_{10}, t_0) = 0\,, \\ T_{12}(x_{10}, t_0) = T_{12}^0\,. \end{gathered} \tag{1.13}$$

The boundary conditions have the form:

$$V_1(x_{10}, t) = V_3(x_{10}, t) = 0, \quad V_2(x_{10}, t) = V_2^0(t), \tag{1.14}$$

$$T_{11}(x_{10},t) = T_{22}(x_{10},t) = T_{33}(x_{10},t) = T_{13}(x_{10},t) = T_{23}(x_{10},t) = 0, T_{12}(x_{10},t) = T_{12}^0(t)\,. \tag{1.15}$$

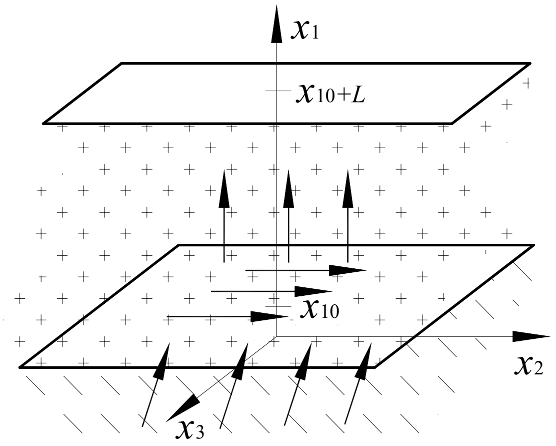

Figure 1

In that case all field values are the functions only of x_1 and the equations (1.1), (1.2), (1.8) – (1.11), (1.5) can be written in the form:

$$\frac{\partial T_{11}}{\partial x_1} - \frac{\partial p}{\partial x_1} + \rho g = \rho \frac{\partial V_1}{\partial t} \tag{1.16}$$

$$\frac{\partial T_{12}}{\partial x_1} = \rho \frac{\partial V_2}{\partial t} \tag{1.17}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x_1} V_1 + \rho \frac{\partial V_1}{\partial x_1} = 0 \tag{1.18}$$

$$\frac{\partial T_{11}}{\partial t} = \alpha_1 \left(\gamma^* + 2\mu^*\right) \frac{\partial u_1}{\partial x_1} + \alpha_2 \left(\gamma^* + 2\mu^*\right) \hat{\delta}_{11} \hat{\delta}_{pq} D_{pq} + \alpha_3 X \hat{\delta}_{pq} D_{pq} \tag{1.19}$$

$$\frac{\partial T_{12}}{\partial t} = \alpha_1 \mu^* \frac{\partial u_2}{\partial x_1} + \alpha_2 \mu^* \hat{\delta}_{12} \hat{\delta}_{pq} D_{pq} \tag{1.20}$$

$$\dot{\delta}_{ij} = \begin{cases} D_{ij} - \hat{\delta}_{ij} \hat{\delta}_{pq} D_{pq}\ , & \text{for } \ \delta_{pq} D_{pq} > 0 \\ D_{ij} & \text{for } \ \delta_{pq} D_{pq} \le 0 \end{cases}, \ i = 1, j = 1,2 \tag{1.21}$$

$$\dot{e} = (1 + e) trD \tag{1.22}$$

The expressions for $\dot{T}_{22}$, $\dot{T}_{33}$ we don't write, because in further calculations we won't use them.

At first let's consider equations (1.21). These equations are nonlinear therefore analytical solution may be found only approximately. We use 1-D approximation for each equation from system (1.21) in linear approximation.

$$\dot{\delta}_{ij} = \begin{cases} D_{ij} - \dfrac{|\delta_{ij}|}{R} D_{ij}\ , & \text{for } \ \delta_{ij} D_{ij} > 0 \\ D_{ij} & \text{for } \ \delta_{ij} D_{ij} \le 0 \end{cases}, \quad \text{i} = 1, \text{j} = 1, 2 \tag{1.23}$$

$$
\delta_{ij} = \begin{cases} \widetilde{E}_{ij}\left(1+\dfrac{\widetilde{E}_{ij}}{2R}\right) & \text{, for } \delta_{ij} > 0,\ D_{ij} > 0 \\ \widetilde{E}_{ij}\left(1-\dfrac{\widetilde{E}_{ij}}{2R}\right) & \text{, for } \delta_{ij} < 0,\ D_{ij} < 0 \qquad i = 1, j = 1, 2 \\ \widetilde{E}_{ij} & \text{, for } \delta_{ij} D_{ij} \le 0 \end{cases} , \tag{1.24}
$$

$$
\dot{\delta}_{ij} = \begin{cases} \left(1+\dfrac{\widetilde{E}_{ij}}{R}\right)D_{ij} & \text{, for } \delta_{ij} > 0,\ D_{ij} > 0 \\ \left(1-\dfrac{\widetilde{E}_{ij}}{R}\right)D_{ij} & \text{, for } \delta_{ij} < 0,\ D_{ij} < 0 \qquad i = 1, j = 1,2 \\ D_{ij} & \text{, for } \delta_{ij} D_{ij} \le 0 \end{cases} , \tag{1.25}
$$

We obtain following equation:

$$
\frac{\partial}{\partial x_1}\left(A_2 \frac{\partial V_2}{\partial x_1}\right) - \rho \frac{\partial^2 V_2}{\partial t^2} = -\frac{\partial v_2}{\partial t}\left(\frac{\partial \rho}{\partial x_1} V_1 + \rho \frac{\partial V_1}{\partial x_1}\right) \tag{1.26}
$$

From (1.38) it follows:

$$
\frac{\partial V_1}{\partial x_1} = \frac{d}{dt}\left[\ln(1+e)\right] \ \Rightarrow \ V_1 = C \ln \frac{1+e}{1+e_0} \tag{1.27}
$$

Substituting (1.27) in (1.26) find

$$
\frac{\partial}{\partial x_1}\left(A_2 \frac{\partial V_2}{\partial x_1}\right) - \rho \frac{\partial^2 V_2}{\partial t^2} = -\frac{\partial V_2}{\partial t}\left(\frac{\partial \rho}{\partial x_1} C(x_1) \ln \frac{1+e}{1+e_0} + \rho \frac{d \ln(1+e)}{dt}\right) \tag{1.28}
$$

In case when $e \ll 1$ we obtain equation for zeroth approximation:

$$
\frac{\partial}{\partial x_1}\left(A_2 \frac{\partial V_2}{\partial x_1}\right) - \rho \frac{\partial^2 V_2}{\partial t^2} = 0 \tag{1.29}
$$

We obtain the differential equation for V_1

$$
\frac{\partial}{\partial x_1}\left(A_1 \frac{\partial V_1}{\partial x_1}\right) - \rho \frac{\partial^2 V_1}{\partial t^2} = -\frac{\partial}{\partial x_1}\left(B_1 \frac{\partial V_2}{\partial x_1}\right) + \left(\frac{\partial \rho}{\partial x_1} V_1 + \rho \frac{\partial V_1}{\partial x_1}\right)\left[p + \frac{\partial V_1}{\partial t}\right] + \frac{\partial^2 p}{\partial x_1 \partial t} \tag{1.30}
$$

For the function $p(x,t)$ we write the equation analogous [6]:

$$\frac{\partial p_e}{\partial t}=-\frac{(p_e+p_a)(1+e)}{e(1-S_r)}\frac{\partial V_1}{\partial x_1}-V_1\frac{\partial p_e}{\partial x_1} \qquad (1.31)$$

At zeroth approximation of (1.30) obtain:

$$\frac{\partial}{\partial x_1}\left(A_1\frac{\partial V_1}{\partial x_1}\right)-\rho\frac{\partial^2 V_1}{\partial t^2}=-\frac{\partial}{\partial x_1}\left(B_1\frac{\partial V_2}{\partial x_1}\right),\ p_e=p_e(x) \qquad (1.32)$$

2. FINDING AND SOLVING OF TRANSFER EQUATIONS IN GENERAL CASE

Consider the equation (1.29). Coefficients A_2, p are functions of t, x_1, we apply the ray method for the solving this equation. Write the solution of the equation in the form of ray series:

$$V_2(x,t)=\sum_{n=0}^{\infty}V_2^{(n)}(x_1)\cdot f_n(t-\psi(x_1)) \qquad (2.1)$$

$$f_n'(\xi)=\frac{df_n(\xi)}{d\xi}=f_{n-1}(\xi) \qquad (2.2)$$

where $\psi(x_1)$ is the eikonal; $V_2^{(n)}$ are the functions to be found; t is the current time; x_1 is the spatial coordinate.

Substituting (2.1) in (1.29) obtain:

$$\sum_{n=0}^{\infty}\left\{\left[(\psi')^2A_2-\rho\right]\cdot V_2^{(n)}f_{n-2}-\left[2A_2\psi'V_2^{(n)'}+(A_2\psi''+A_2'\psi')V_2^{(n)}\right]f_{n-1}+\left[A_2V_2^{(n)''}+A_2'V_2^{(n)'}\right]f_n\right\}$$

where the sign,, ′ “ is derivative.

For $n<0$ must be $V_2^{(n)}=0$ because the coefficient by f_0 is equal to zero.

From (2.2) it follows

$$\left[(\psi')^2A_2-\rho\right]\cdot V_2^{(n+1)}-2A_2\psi'V_2^{(n)'}-(A_2\psi''+A_2'\psi')V_2^{(n)}+A_2V_2^{(n-1)''}+A_2'V_2^{(n-1)'}=0,$$

$n=-1,0,1,...$ (2.3)

For $n=-1$ $V_2^{(n)}\equiv 0$, but $V_2^{(0)}\neq 0$, then obtain from (2.3)

$$\left(\psi'_{(x_1)}\right)^2=\frac{\rho(x)}{A_2(x)}=C^{-2}(x), \qquad (2.4)$$

where $C(x)$ is the velocity of the wave.

The equation (2.4) is called eikonal equation. This equation is applied in geometrical acoustic (seismology).

The solution of the equation (2.4) has the form:

$$\psi(x_1) = \psi_{(x,0)} \pm \int_{x_0}^{x_1} \frac{dx_1}{C(x_1)} \tag{2.5}$$

The function $\psi(x_1)$ is the phase of wave or time of wave arrival in the point x_1. In the formula (2.5) we take sign " + " for waves in the direction x_1, sign " – " for waves in the direction $-x_1$.

3. COMPARISON OF RESULTS OF WAVE PROPAGATION IN HIPOPLASTIC MEDIUMWITH AND WITHOUT REGISTRATION OF INTERGRANULAR STRAIN

The velocities of quasi-longitudinal and quasi-transverse waves have the forms:

$$\begin{aligned} C_e^2 &= \frac{A_0}{\rho} = C_e^{2(0)}\left[\alpha_1 + \alpha_2(\operatorname{sign} D_{11})\frac{\tilde{E}_{11}}{R}\right] + \alpha_3\frac{X}{\rho}(\operatorname{sign}\tilde{E}_{11}) \\ C_t^2 &= \frac{A_2}{\rho} = C_t^{2(0)}\left[\alpha_1 + \alpha_2(\operatorname{sign} D_{11})\frac{\tilde{E}_{12}}{R}\right] \end{aligned} \tag{3.1}$$

The expression for $C_e^{2(n)}$, $C_t^{2(n)}$ follows from (3.1). We can write:

$$\begin{aligned} C_e^2 &= C_e^{2(n)} + C_e^{2(i)} \\ C_t^2 &= C_t^{2(n)} + C_t^{2(i)} \end{aligned} \tag{3.2}$$

$$\begin{aligned} C_e^{2(i)} &= C_e^{2(0)}\alpha_2(\operatorname{sign} D_{11})\frac{\tilde{E}_{11}}{R} + \alpha_3\frac{X}{\rho}(\operatorname{sign}\tilde{E}_{11}) \\ C_e^{2(i)} &= C_e^{2(0)}\alpha_2(\operatorname{sign} D_{12})\frac{\tilde{E}_{12}}{R} \end{aligned} \tag{3.3}$$

From (3.2), (3.3) it follows that values of the velocities depend on the sign of D_{11}, D_{12}, $\tilde{E}_{11}$, α_2, α_3.

In paper [6] are given numerical meaning for the coefficients:

$R = 1\cdot 10^{-4}$; $m_R = 5{,}0$; $m_T = 2{,}0$; $\beta_r = 0{,}5$; $\chi = 6{,}0$;

then in formulas (3.1), (3.2), (3.3) we obtain $\alpha_1 > 0$, $\alpha_2 < 0$, $\alpha_6 > 0$.

Therefore by loading quasi-longitudinal wave propagates with the velocity which is less then velocity for elastic skeleton, by unloading it may be different situation for concrete date. The velocity quasi-transverse wave is less than the velocities of transverse wave in elastic skeleton.

4. CONCLUSIONS

1. The equations for wave propagation in a hipoplastic medium are obtained with a registration of intergranular strain.
2. These equations describe the evolution of plane disturbances in case when initial state and the density of medium are inhomogeneous. Therefore the dynamic equations are coupled with each other.
3. In case when void ratio $e << 1$ the dynamic equations are obtained for the quasi-longitudinal and quasi-transverse waves, moreover for transverse component the dynamic equation is closed and for longitudinal component the dynamic equation contains the transverse component.
4. The solutions of the dynamic equations are obtained in general analytical form with the help of the ray method. Ray method allows finding the solutions of the dynamic equations in hipoplastic medium with intergranular strain. These solutions calculate along ray trajectories in a vicinity of wave surface.
5. The velocity of quasi-longitudinal wave in hipoplastic medium with intergranular strain is less than in elastic skeleton by loading and it may be less or bigger by unloading (it depends on conditions of concrete problem). The velocity of quasi-transverse wave in hipoplastic medium with intergranular strain is less than in elastic skeleton.
6. The obtained results in case of homogeneous initial state and constant density of hipoplastic medium are analogous to the results that are obtained by other method.

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