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CAN WE GUARANTEE THE ENHANCEMENT OF ACCURACY IN TIME INTEGRATION ANALYSES WITH SMALLER NONLINEARITY TOLERANCES?

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Abstract

In presence of nonlinearity, more accurate responses are not always obtainable with smaller time steps. From a mathematical point of view; similar to time steps, nonlinearity tolerances are algorithmic parameters. Hence, it is worth investigating the possibility of increasing the accuracy by time integration with smaller tolerances. This is the subject of the study here. As the result, smaller tolerances can not guarantee more accuracy for time integration analyses.

INTRODUCTION

In order to analyze the behaviour of structural systems in the general case of systems and behaviours, it is broadly accepted to follow the procedure below;

1- Defining the structural model, i.e., determining the parameters affecting the structural behaviour (according to the purpose of the analysis).
2- Stating the dynamic equilibrium (equation of motion) of the structural model, the corresponding initial conditions, and probably the existing constraints, by setting the mathematical model corresponding to the model obtained in stage 1.
3- Discretizing the mathematical model obtained in the stage 2 above by methods such as FEM (Finite Elements Method), BEM (Boundary Elements Method), etc., and arriving at a mathematical model discretized in space and continuous in time.
4- Analyzing the semi-discretized model obtained in stage 3 by one of the well-known time integration methods, e.g., the average acceleration method of Newmark [12].

In view of the last two stages, the problem to be analyzed is stated below [3,8,17],

$$\begin{array}{lll} \mathbf{M}\ddot{\mathbf{u}}(t)+\mathbf{f}_{\text{int}}=\mathbf{f}(t) & 0\le t<T, & \\ \text{Initial Condition :} & \mathbf{u}(0)=\mathbf{u}_0\,,\quad \dot{\mathbf{u}}(0)=\dot{\mathbf{u}}_0\,,\quad \mathbf{f}_{\text{int}}(0)=\mathbf{f}_{\text{int}_0}\,, & \\ \text{Additional Constraint s:} & \mathbf{Q}\,. & \end{array} \tag{1}$$

In Eq. (1), $\mathbf{M}$ represents the mass matrix; $\mathbf{f}_{\text{int}}$ and $\mathbf{f}(t)$ respectively express the vectors of internal forces and external excitations; $\mathbf{u}(t)$, $\dot{\mathbf{u}}(t)$, and $\ddot{\mathbf{u}}(t)$ denote the vectors of displacement, velocity, and acceleration, respectively; $\mathbf{u}_0$, $\dot{\mathbf{u}}_0$, and $\mathbf{f}_{\text{int}_0}$ imply the initial status, and $\mathbf{Q}$ stands for the probable restricting constraints, e.g., in problems involved in impact or elastic-plastic behaviour [10,24]; all with respect to the degrees of freedom set for the system under consideration. Time integration of Eq. (1) can not be carried out exactly; the reason is also the complexities of the exact formulation essential in time integration of MDOF (Multi-Degree-of-Freedom) systems involved in nonlinearities [5,7]. Consequently; since the pioneering methods of J.L.Houbolt [9] and N.Newmark [12] (see [23]), many methods are proposed based on different approximations [2,6,9,11,23]. In Presence of nonlinearity, besides the inexact formulation of the integration methods, errors due to inexact nonlinearity solutions affect the accuracy [20]. The errors due to the approximate formulation can be changed by similarly decreasing/increasing the sizes of all time steps along the time interval T in Eqs. (1) (according to a parameter namely Δt). Nevertheless, the accepted way to control the errors of nonlinearity solutions is making use of nonlinearity tolerances, i.e. $\bar{\delta}$, in iterative procedures; in nonlinearity-detected time steps, iterative nonlinearity solution methods are being implemented [4], ending with

$$\begin{array}{ll} i=1,2,3,\ldots k-1 & \|\delta_i\|>\bar{\delta}\,, \\ & \|\delta_k\|\le\bar{\delta}\,. \end{array} \tag{2}$$

In Eq. (2), δ_i implies the error of the nonlinearity solution. e.g., the out of balance force; $\bar{\delta}$ is the corresponding nonlinearity tolerance; and $\|\;\|$ stands for an arbitrary norm [13]. Equation (2) leads to

1- $\bar{\delta}$ controls the accuracy of the nonlinearity solution; and hence, similar to Δt, $\bar{\delta}$ is an algorithmic parameter controlling the accuracy of the integration analysis,
2- Even with smaller $\bar{\delta}$, arriving at smaller $\|\delta_k\|$ can not be guaranteed. Therefore, different from Δt, which can reliably control the errors originated in the approximate formulation, $\bar{\delta}$ is not a reliable controller for $\|\delta_i\|$.

In view of the two points above, the trend of error changes with respect to nonlinearity tolerances might even be worse than the error changes with respect to time step size. This signifies the importance of the study in this paper. Brief theoretical and numerical studies are carried out in the next two sections.

THEORY

In view of the well established linear SDOF error formulation in the literature [2]; denoting the displacement and velocity with v and $\dot{v}$ respectively, and implementing the definition below,

$$\mathbf{e}_i = \begin{Bmatrix} v_i \\ \dot{v}_i \end{Bmatrix} - \begin{Bmatrix} v(t_i) \\ \dot{v}(t_i) \end{Bmatrix}, \tag{3}$$

as the vector of deviation of the approximate responses $\{v_i \;\; \dot{v}_i\}^T$ (T as the right superscript implies matrix transposition) from analytical responses $\{v(t_i) \;\; \dot{v}(t_i)\}^T$, at t_i; $\mathbf{e}_i$ can be expressed as

$$\mathbf{e}_i = \mathbf{A}_i \mathbf{e}_{i-1} + \tau_i . \tag{4}$$

In Eq. (4), the $\mathbf{A}_i$ and τ_i respectively stand for the amplification matrix [2] and the vector of local errors [22] at the time step ending to t_i. Starting from Eq. (3), and assuming precise initial conditions, the step-wise modal error formulation considering nonlinear behaviour is recently proposed [20,21] as,

$$\begin{Bmatrix} {}_i\mathbf{e}_i^1 \\ {}_i\mathbf{e}_i^2 \\ \vdots \\ {}_i\mathbf{e}_i^n \end{Bmatrix} = \sum_{j=1}^{i} \left(\prod_{k=1}^{i-j} {}_{k+j}\boldsymbol{\Psi}_{k+j+1/k+j} \begin{bmatrix} \mathbf{A}_{k+j}^1 & & & \mathbf{0} \\ & \mathbf{A}_{k+j}^2 & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{A}_{k+j}^n \end{bmatrix} \right) \left(\begin{Bmatrix} \tau_j^1 \\ \tau_j^2 \\ \vdots \\ \tau_j^n \end{Bmatrix} + \begin{Bmatrix} {}^{NL}\Delta_j^1 \\ {}^{NL}\Delta_j^2 \\ \vdots \\ {}^{NL}\Delta_j^n \end{Bmatrix} \right) \tag{5}$$

(and later resulted in a formulation for the errors of arbitrary component of the response of nonlinear semi-discretized equations of motion [19]). In Eq. (5); the new variable $\boldsymbol{\psi}$ represents the matrix converting the modal descriptions at two adjacent time steps [20,21], the right and left subscripts respectively stand for the time step and the time station, the right superscript is an identifier for the vector member, and finally ${}^{NL}\Delta$ implies the effects of the δ_k in Eq. (2) on the errors under consideration. Nonlinearity tolerances ($\bar{\delta}$) affect Eq. (5) only implicitly via Eq. (2). In addition, Eq. (2) is an inequality and, as discussed in the previous section, can not reliably control nonlinearity solutions. Hence, it seems reasonable to conclude that smaller tolerances can not guarantee less error. Therefore, the response to the question in the title of this paper is negative, i.e., with only decreasing $\bar{\delta}$, achieving more accuracy from time integration analyses is nothing more than probable. However, due to reasons such as the secondary role of the nonlinearity tolerances in time integration analysis (compared to the integration step size), the numerical value of the phenomenon explained above might be trivial. Also, in order to clarify this ambiguity; in the next section, besides studying whether we have correctly gave a negative response to the main question of this paper, error changes with respect to the time step size will be compared with those with respect to the nonlinearity tolerances.

NUMERICAL ILLUSTRATION

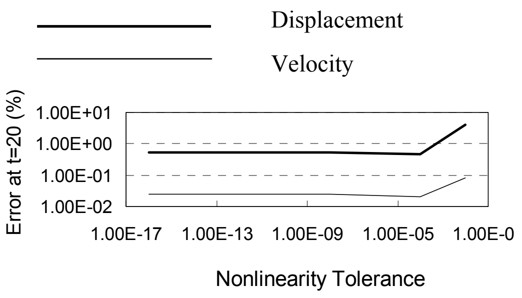
To study the changes of errors with respect to the nonlinearity tolerance and compare these changes with those with respect to the time step size; it is reasonable to first study the special examples for which the changes of errors with respect to the time step size is reported in the literature. In view of this consideration, an appropriate example is as noted below [15,18,21],

$$\begin{gathered} M\left(\ddot{u}+\ddot{u}_g(t)\right)+f_{\text{int}}=0 \;,\quad u(t=0)=\dot{u}(t=0)=f_{\text{int}}(t=0)\;,\quad T:\;\; 0\le t<10 \\ M=1\times10^{5}\;\;\text{kg},\quad C=0\;\;\text{N s/m},\quad f_{\text{int}}=C\,\dot{u}+f_s \\ f_s=\text{Linear Elastic}-\text{Plastic with Kinematic Hardening} \\ K=1\times10^{5}\;\;\text{N/m},\; K_{plastic}=1\times10^{4}\;\;\text{N/m},\; u_y=1\times10^{-2}\;\;\text{m} \\ \ddot{u}_g(t)=\text{N}-\text{S acceleration component of the El Centro strong motion }[5]. \end{gathered} \tag{6}$$

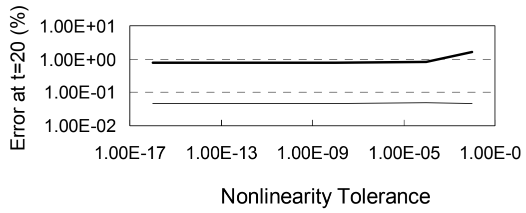
Analyzing Eq. (6) with the average acceleration method of Newmark [12] (average acceleration is the method recommended in the literature for problems involved in nonlinearity [1]) with time step sizes, $\Delta t = 0.02$ seconds, nonlinearity tolerances $\bar{\delta} = 0.01$, and other parameters similar to the parameters in the literature [15,20], the analysis is then repeated with only changing the nonlinearity tolerances to the new tolerances noted below

$$\bar{\delta} = 10^{-4}, 10^{-8}, 10^{-16}. \tag{7}$$

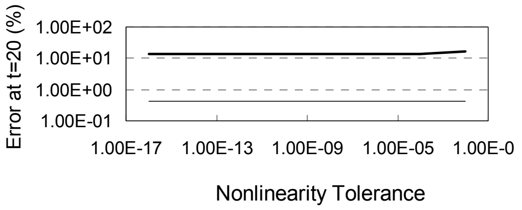
Reporting the errors [14] of the obtained responses in Fig. 1 justifies the negative response to the question in the title of this paper. Meanwhile, comparing Fig. 1 with Fig. 2 [21] clearly demonstrates the fact that, as explained theoretically in the previous section, similar to the decrease of time steps, implementing smaller nonlinearity tolerances, can not guarantee enhanced accuracy. This is in complete agreement with the other researches reported in the literature [16,21]. It is also instructive to note that, in view of Figs. 1 and 2, the changes of errors with respect to the nonlinearity tolerances can be more reliable than those changes with respect to the time step sizes.



a) Average Acceleration [12]

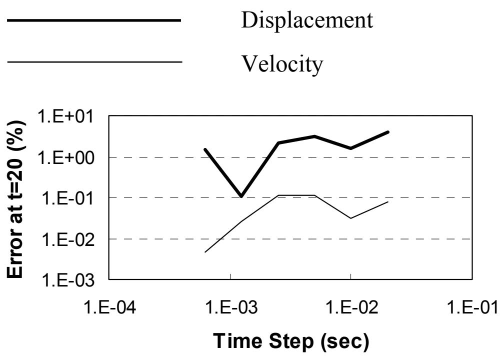


b) Central Difference [23]

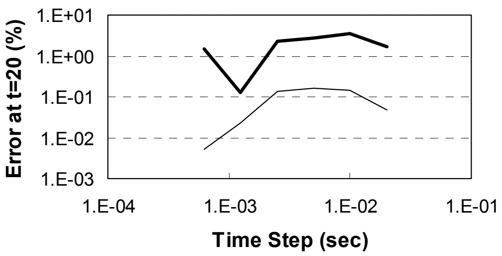


c) HHT [23]

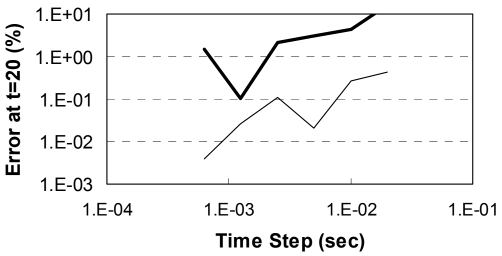
Figure 1 – Changes of errors with respect to nonlinearity tolerance, for Eq. (6), when $\Delta t = 0.02$



a) Average Acceleration [12]



b) Central Difference [23]



c) HHT [23]

Figure 2 – Changes of errors with respect to time step size, for Eq. (6), when $\overline{\delta} = 0.01$

CONCLUSION

Time integration of nonlinear semi-discretized equations of motion with smaller nonlinearity tolerances can not guarantee more accuracy. The trends of errors changes with respect to the nonlinearity tolerances might be more predictable compared to those with respect to time step sizes. Further research is being recommended.

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