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MODAL DETECTION FOR NON-UNIFORM BEAMS USING CONTINUOUS AND DISCRETE STRAIN SENSORS

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Abstract
Surface mounted strain based sensors provide a more practical and cost effective solution for measuring plate vibration for in-field applications such as Active Structural Acoustic Control (ASAC). When considering non-uniform plates the orthogonality condition (OC) based on the plate kinetic energy cannot be used, and an approximate strain energy based OC has been developed for beams using with continuous strain sensors, such as polyvinylidene fluoride (PVDF) [3]. The OC is converted for use with discrete sensors and compared with the Least Mean Squares method (LMS) method. It is found that using the OC as a basis for modal detection is not successful for strain sensing and the LMS method is better but still not as good as when using displacement detection. The consequence of using strain sensing in general is discussed, and also the more detrimental aliasing effect for discrete sensors when applied to non-uniform beams.

INTRODUCTION

Active Structural Acoustic Control (ASAC) [1] is a technique used to control sound radiation from panels. It reduces the sound radiated by measuring and controlling the radiation modes of the panel. These can be represented by combinations of structural modes, and are often measured using polyvinylidene fluoride (PVDF) strain sensors. Such systems have been successfully developed for uniform rectangular plates however in practice radiating surfaces are more complex. With the aim of designing sensors for more realistic panels, the measurement of structural modes for non-uniform beams using both continuous and discrete strain sensors is studied. A strain energy based orthogonality condition has been developed for use with continuous sensors [3] and a discrete version is compared with the Least Mean Squares (LMS) method for discrete sensors and applied to a non-uniform beam.

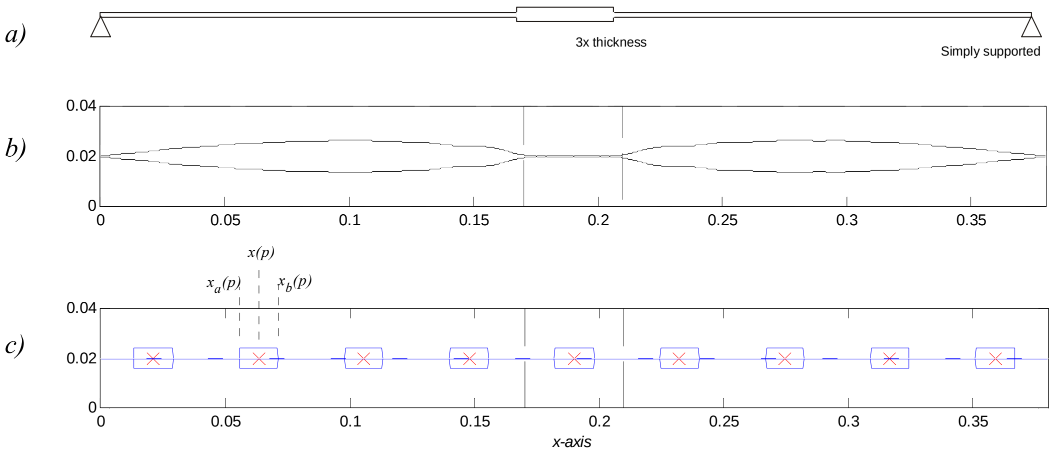

Figure 1 – a) The non-uniform simply supported beam considered of dimension 380×40mm and 2mm/6mm thickness, b) plan view of PVDF sensor for mode (3,1), c) plan view of nine piezoelectric sensor patches.

ORTHOGONALITY FOR STRAIN SENSORS

The orthogonality condition for plate vibration normally encountered is defined for displacement (or temporal derivatives) and is not valid for strain measuring sensors. For *uniform* plates with simply supported, clamped or free boundary conditions the second spatial derivative modeshapes are either the same as for displacement or form a set of mutually orthogonal modeshapes and do not require special consideration.

Kinetic energy based orthogonality condition

Modal orthogonality is normally derived from the independence of the kinetic energy of each mode [2]. This provides a simple expression that is directly applicable to systems measuring the transverse displacement or temporal derivatives. The kinetic energy of a plate when two modes m and n exist simultaneously must be the sum of the energies when the modes exist separately, by energy conservation. So

$$KE_{m+n} = \frac{1}{2}\int_S m'' v_m^2 \varphi_m^2 \,\mathrm{d}S + \frac{1}{2}\int_S m'' v_n^2 \varphi_n^2 \,\mathrm{d}S = \frac{1}{2}\int_S m'' \left(v_m\varphi_m + v_n\varphi_n\right)^2 \mathrm{d}S \qquad (1)$$

where m'' is the plate mass distribution, and v_m the transverse velocity amplitude of the m^{th} mode shape, φ_m . This implies that the cross-product must be zero when $m \neq n$, giving the orthogonality condition

$$\int_S m'' \varphi_m \varphi_n \,\mathrm{d}S = \begin{cases} 0 & m \neq n \\ \Lambda_m & m = n \end{cases}, \qquad (2)$$

where Λ_m is the modal mass. For uniform plates where the mass distribution is constant the mass term can be taken outside the integral and excluded from the definition. However, for non-uniform plates (non-uniform thickness or density) it is imperative that the mass distribution be applied as a weighting applied to the measured distribution of plate surface motion.

When using discrete sensors, the beam is divided up into equal elements and it is assumed that the measurement about the centre point is representative of the entire element. This imposes a limit defined by the number of points and the structural wavelength to prevent aliasing effects. Eqn (2) can be discretised as

$$\sum_{p=1:P} m_e \varphi_m(p)\varphi_n(p) = \begin{cases} 0 & m \neq n \\ \Lambda_m & m = n \end{cases} \quad \text{for} \quad m,n \leq P \ . \tag{3}$$

where m_e is the elemental mass and p the measurement point, as shown in Figure 1. Two versions of m_e can be considered, one using the mass distribution at the measurement point and another using the mass over the entire elemental area. Here the latter is used.

Strain energy based orthogonality condition

The use of sensors applied a plate to measure surface strain are commonly used to detect flexion. When are applied to a plate in bending, the surface strain is equivalent to the second order spatial derivative in this direction if the plate is plane-symmetric about the mid plane parallel to the surfaces. The strain energy of a plate in bending is also defined in terms of second order spatial derivatives and this allows an orthogonality condition to be defined in the same terms as measured by strain sensors.

The maximum potential energy, U_p, (through flexural strain) of a plate of dimensions (L_x, L_y) undergoing vibration may be adapted for application to non-uniform plates as [4]

$$U_P = \frac{1}{2}\int_0^{L_y}\int_0^{L_x} D''(x,y)\left[\left(\frac{\partial^2 W}{\partial x^2}\right)^2 + \left(\frac{\partial^2 W}{\partial y^2}\right)^2 + 2v(x,y)\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2} + 2\left(1 - v(x,y)\right)\left(\frac{\partial^2 W}{\partial x \partial y}\right)^2\right] \mathrm{d}x\mathrm{d}y \,, \tag{4}$$

where W is the transverse plate displacement, v is Poisson's ratio and D'' is the plate bending rigidity area density. The beam system considered here is essentially a 1-D beam system, and it has been shown previously [3] that for the first ten structural modes (using a 2-D model) 95% of the strain energy is due to the left most bracketed term in eqn (4). So the strain energy in the beam can be approximated as

$$U_P \approx \frac{1}{2}\int_0^{L_y}\int_0^{L_x} D''(x,y)\left(\frac{\partial^2 W}{\partial x^2}\right)^2 \mathrm{d}x\mathrm{d}y \ . \tag{5}$$

It was also shown in [3] that this leads to a simple approximate orthogonality condition, which for a non-uniform homogenous beam (constant physical properties and a variable beam thickness, $h_p(x)$) is

$$\int_0^{L_x} h_p^3(x)\frac{\partial^2 \phi_m}{\partial x^2}\frac{\partial^2 \phi_n}{\partial x^2}\mathrm{d}x \approx \begin{cases} 0 & m \neq n \\ \Lambda_s^h & m = n \end{cases} . \tag{6}$$

As with the consideration of kinetic energy, the strain based energy orthogonality condition (6) can also be discretised as

$$\sum_{p=1:P} s_e \varphi''_m(p)\varphi''_n(p) = \begin{cases} 0 & m \neq n \\ \Lambda_s & m = n \end{cases} \quad \text{for} \quad m,n \leq P \ , \tag{7}$$

where s_e represents the elemental strain energy. Again, this could be considered as the value of the plate bending rigidity area density at the measurement point, or the value integrated over the area of the element. The latter is implemented here.

Least Mean Squares (LMS) method

An alternative model detection strategy is the LMS method. The amplitudes of the modeshape contained in the modal matrix (**Φ**) are adjusted to best describe the displacement field, **W**, at points p. The optimum modal amplitudes are defined as

$$\mathbf{A}_{\mathbf{opt}}(\omega) = (\mathbf{\Phi}^{\mathrm{T}}\mathbf{\Phi})^{-1}\mathbf{\Phi}^{\mathrm{T}}\mathbf{W}(\omega) \ . \tag{8}$$

To best describe the *kinetic energy* of the system the mass distribution can be included via a diagonal mass matrix, $\mathbf{M_s}$. Then the optimum modal amplitudes are

$$\mathbf{A}_{\mathbf{opt}}(\omega) = (\mathbf{\Phi}^{\mathrm{T}}\mathbf{M_s}\mathbf{\Phi})^{-1}\mathbf{\Phi}^{\mathrm{T}}\mathbf{M_s}\mathbf{W}(\omega) \ . \tag{9}$$

The LMS method can also be applied to find the optimum modal amplitudes to best describe the second spatial derivative of displacement at the measurement points p:

$$\mathbf{A}_{\mathbf{opt}}(\omega) = (\mathbf{\Phi}''^{\mathrm{T}}\mathbf{\Phi}'')^{-1}\mathbf{\Phi}''^{\mathrm{T}}\mathbf{W}''(\omega) \ , \tag{10}$$

where $\mathbf{\Phi}''$ is the second spatial derivative modal matrix and $\mathbf{W}''$ is the second spatial derivative of the displacement field. The fit can also be made using the strain energy of the beam by including the diagonal bending rigidity matrix $\mathbf{D_s}$:

$$\mathbf{A}_{\mathbf{opt}}(\omega) = (\mathbf{\Phi}''^{\mathrm{T}}\mathbf{D_s}\mathbf{\Phi}'')^{-1}\mathbf{\Phi}''^{\mathrm{T}}\mathbf{D_s}\mathbf{W}(\omega) \ . \tag{11}$$

CONTINUOUS AND DISCRETE STRAIN SENSORS

Continuous PVDF sensor

The response of a PVDF sensor applied to the surface of a uniform plate of dimensions (L_x, L_y) is an output electrical charge, $q(t)$, for a non-uniform thickness beam is [3]

$$q(t) \approx e^{j\omega t}\,\frac{1}{2}\int_0^{L_x}\int_{y_s-bS(x)}^{y_s+bS(x)} h_p(x)\left(e_{31}^0\,\frac{\partial^2 W}{\partial x^2} + e_{32}^0\,\frac{\partial^2 W}{\partial y^2}\right)\mathrm{d}y\mathrm{d}x \ . \tag{12}$$

W the displacement field of the plate, e_{31}^0 and e_{32}^0 are the charge constants relating to the x and y directions respectively. h_p is the thickness of the plate. This describes a

PVDF sensor in the form of a strip in the *x*-direction, located at $y = y_s$, and having a width profile function of $S(x)$ and a corresponding scaling factor b. When applying the PVDF sensor to non-uniform surface the height profile provides a natural spatial sensor weighting in addition to $S(x)$. For a 1-D consideration, the response of the PVDF sensor in (12) can also be simplified so that

$$q(t) \propto \int_0^{L_x} S(x) h_p(x) \frac{\partial^2 W}{\partial x^2} \mathrm{d}x \ . \tag{13}$$

Discrete piezoelectric sensor patch

Piezoelectric material responds to strain in the same way as PVDF. For a piezoelectric rectangular patch applied to the plate surface, see Figure 1, its response by eqn (13) is proportional to the difference in bending at the limits of the patch. If the patch is small compared to the highest structural wavenumber measured then the output is proportional to the second spatial derivative about its mid-point. So

$$q(t) \ \propto \ h_p(p) \left[\left. \frac{\partial W}{\partial x} \right|_{x_a(p)} - \left. \frac{\partial W}{\partial x} \right|_{x_b(p)} \right] \ \propto \ h_p(p) \left| x_a(p) - x_b(p) \right| \left. \frac{\partial^2 W}{\partial x^2} \right|_{x(p)} \ . \tag{14}$$

APPLYING STRAIN SENSORS FOR MODAL DETECTION

Results are now given using a numerical model that uses a Rayleigh-Ritz solution to solve for the modeshapes of the non-uniform beam. The output of the strain sensors is achieved by integrating the second order spatial derivative of displacement over the sensor area [4]. The modal amplitudes are normalised so that the value of the below-resonance residual of each mode is proportional to $1/\omega_n$. This provides a general modal excitation diminishing with modal order and is not dependent upon a specific actuator location. Modes up to the 40th mode, with resonant frequency around 11kHz, are included in the structural model as this necessary to accurately model the strain sensor, and discussed further below.

Continuous PVDF sensor

Figure 2 shows the result of applying a series of PVDF sensors strips to the beam, as shown in Figure 1. It is seen that the application of the sensor with only the natural height profile provides errors in the modal detection a high frequencies and also in the low frequency residue region. Applying the correct weighting (eqn (13): $S(x) = h_p^2(x)$) to the sensor it is seen that better detection is achieved if the sensor does not have the cross-axis response ($e^0_{32} = 0$). For each mode good detection over a 60dB range is seen. However, the response to cross-axis bending reduces the success dramatically. In practice it is thought that the strain transfer in the cross-axis direction would be reduced, but this needs to be verified experimentally.

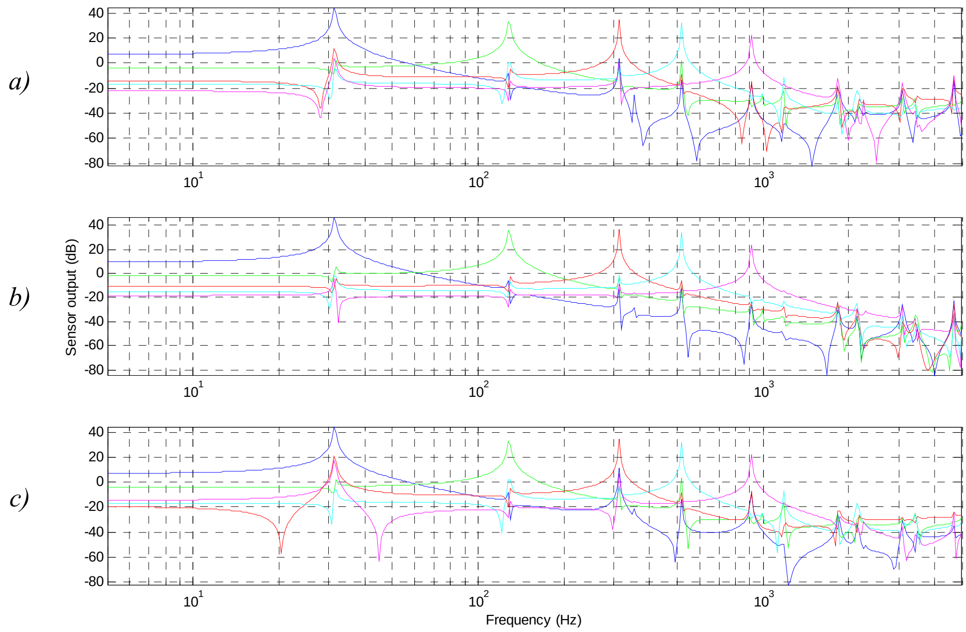

Figure 2 – The response of continuous PVDF sensor. a) only the natural height weighting function, b) using the bending rigidity weighting function but with no cross-axis sensor sensibility, c) using the bending rigidity weighting function and full PVDF sensor model.

Discrete sensors

The use of displacement measurements is used as a benchmark for discrete measurements. It was found that using the orthogonality condition for displacement sensing was fairly successful but was not so for strain sensing. Using the orthogonality condition as a basis for modal detection was found to be sensitive to small errors in the measurement points (<0.1%) and is not considered further here due to limitation of space. The result of using the LMS method for detecting the first five modes using nine measurement points is shown in Figure 3. It is seen that this method is not sensitive to small errors in the points *p* and that the use of the mass weighting function helps to reduce errors in the high frequency "roll-off" region of the modal responses. However errors still persist at higher frequency due to the residues of higher order modes not being represented in the modal matrix. Lastly, good detection of the first five modes is achieved with a full modal matrix containing the same number of modes as measurement points, although only five modes are required the amplitudes of the other modes are extracted and shown. The remaining errors are due to aliasing from higher order modes. Thus, good detection of the first five modes is achieved to about 2kHz.

Figure 4 shows the corresponding results using strain sensors in place of displacement sensing. The weighting matrix, $\mathbf{D_s}$, has no significant effect when the modal matrix only contains the five modes required. Better detection results with a full modal matrix but is only accurate to about 250Hz. In general the detection of residues from higher order mode at higher frequencies is seen in the levelling out of the roll-off of the modes.

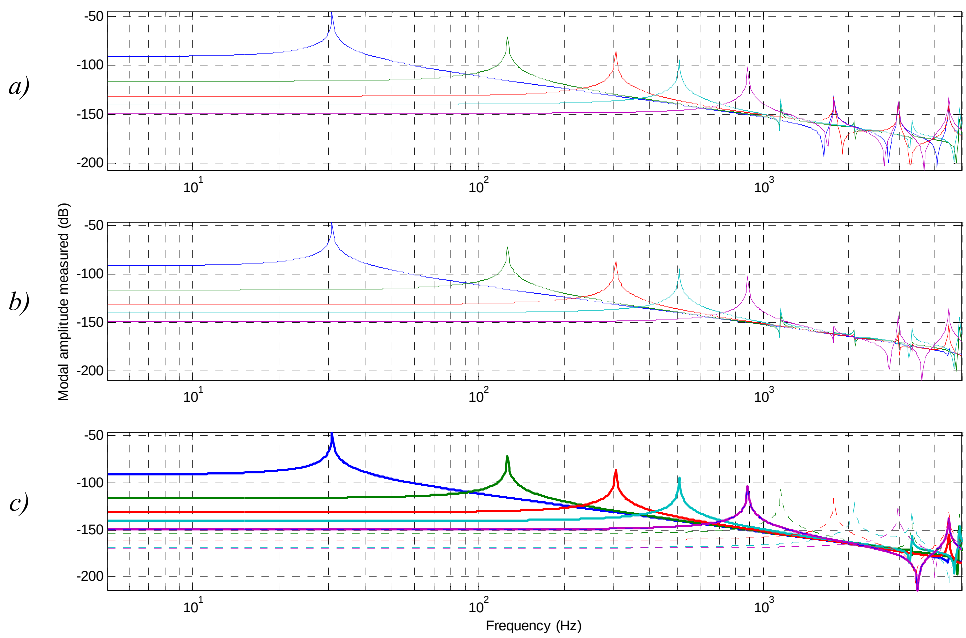

Figure 3 – The first five modal amplitudes detected using the LMS method with nine discrete displacement sensors. a) **Φ** *containing only five modes and without* $\mathbf{M_s}$*, b) as a) but with* $\mathbf{M_s}$*, c)* **Φ** *containing nine modes (*$\mathbf{M_s}$ *has no effect).*

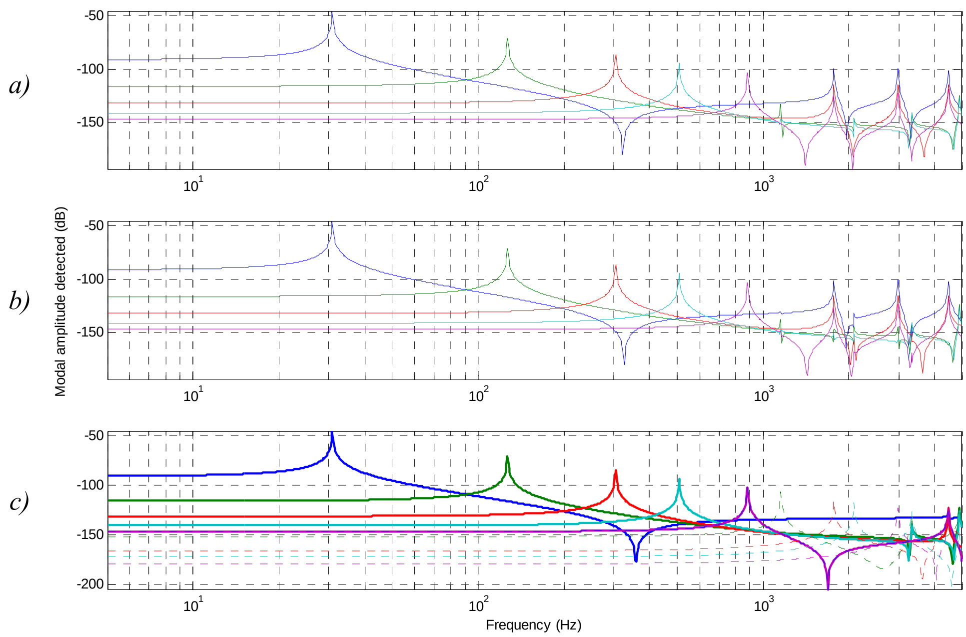

Figure 4 – The first five modal amplitudes detected using the LMS method with nine discrete strain sensors. a) **Φ** *containing only five modes and without* $\mathbf{D_s}$*, b) as a) but with* $\mathbf{D_s}$*, c)* **Φ** *containing nine modes (*$\mathbf{D_s}$ *has no effect).*

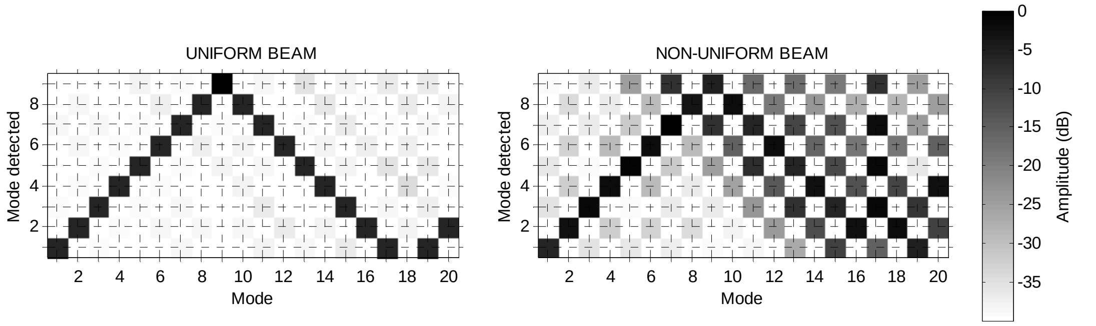

Figure 5 – Mode detected against the modal order for a uniform and non-uniform beam with nine discrete displacement sensors. The grey bar indicates the amplitude of the components.

MODAL DETECTION FOR STRAIN SENSORS ON NON-UNIFORM BEAMS

There are two reasons that make modal detection of non-uniform beam using strain sensors difficult. The first is the natural amplification of the higher order modal residues due to performing the detection in the second order derivative of displacement domain. This was discussed and illustrated in [3]. Secondly, for non-uniform beams the aliasing that occurs is not the simple folding effect as shown in Figure 5. For non-uniform beams some aliasing exist between modes below the Nyquist limit. Also the aliasing is more complex with many closely spaced modes alias to the same detected mode. This is compounded by natural amplification of the higher order modal residuals, as discussed. This effect is not included in Figure 5.

SUMMARY

Modal detection for non-uniform beam using continuous strain sensors is impeded by the consequence of detecting in the second order spatial derivative domain. Discrete sensors additionally suffer from a complex and more detrimental form of aliasing which is a problem of spatially sampling a non-uniform system. This is compounded with the previous difficulty. Thus in general, a greater number of discrete sensors are required for strain sensing to achieve the same performance as for displacement sensing.

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