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INVERSE MODEL OF BEAM FOR BOUNDARY CONDITION IDENTIFICATION

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Abstract
The purpose of the study is demonstration of possibility of boundary conditions identification in under-determined problem. The problem is understood as determining four constants necessary for description of the functions of forced vibration amplitudes, from three equations. For this the Singular Value Decomposition (SVD) algorithm is used.
After estimation the function of forced vibration amplitudes, support elasticities coefficients can be calculated from the equations describing the boundary conditions.
Verification of the obtained mathematical model (elastically supported Bernoulli-Euler's beam) was done by: comparing natural frequencies obtained from the analytical and numerical model, and analysis the correlation between the vibration amplitude vectors.

INTRODUCTION

Analysis of the dynamic processes of real objects can be expensive, time-consuming and in certain cases impossible, whereas experiments can be easily carried out on models, which can be used to simulate dynamic response. For this purpose a physical and mathematical model of the object should be built followed by estimation of model parameters and model verification. This process is called identification of mechanical systems.

Here the analysed system is a beam, described by Bernoulli-Euler's model with unknown boundary conditions, modelled by elastic supports. Mathematical model of the boundary conditions is described by equations binding respectively the bending moment and angle of rotation and lateral force and amplitude of vibrations in cross sections, in which the beam is supported.

In order to determine the support elasticities an inverse model of the beam has been created. In the paper the inverse model is understood as determining the functions of amplitudes of forced vibration based on the measurements of amplitudes in several points, and then, support elasticities can be calculated from the equations describing the boundary conditions.

The function of the vibration amplitudes caused by a given force, is described by the equation containing four sought constants. The simplest method of finding the constants is to measure the amplitudes of the vibrations caused by the force of known amplitude and frequency in four points. In order to minimize measuring errors the measurements can be taken in larger number of points and then one of the statistical methods e.g. regression analysis can be used. The biggest problem is encountered when the available number of measurement values is lower than the number of constants to be determined. (so-called under-determined problem [4]). In these cases four constants from three equations can be determined by using decomposition of the main matrix by Singular Value Decomposition (SVD) algorithm [3].

Decomposition by SVD algorithm is also used for determining the inverse models of over-determined systems (with excess information about the system) e.g. in order to find the power of acoustic wave sources [2, 5, 6].

INVERSE MODEL OF THE BEAM

Model of the beam on elastic supports is shown in Fig. 1.

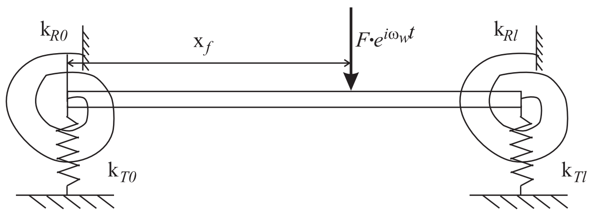


Figure 1: The beam with general boundary conditions

The differential equation of beam vibration has a form:

$$EI \cdot \frac{\partial^4 y(x,t)}{\partial x^4} + \rho A \cdot \frac{\partial^2 y(x,t)}{\partial t^2} = F \cdot \delta(x, x_f) \cdot e^{i\omega_w t} \tag{1}$$

Equation (1) can be solving by separating the variables i.e.: $y(x,t) = X(x) \cdot T(t)$. In steady-state, the ,,time” equation may be expressed as: $T(t) = 1 \cdot e^{i\omega_w t}$

In this case the differential equation of ,,space” variable take the form:

$$X^{(4)}(x) - \lambda^4 \cdot X(x) = F/EI \cdot \delta(x, x_f) \tag{2}$$

where: $\lambda^4 = \omega_w^2 \cdot \rho A/EI$,
and its solution is function (3):

$$\begin{aligned} X(x) &= P\cosh\lambda x + Q\sinh\lambda x + R\cos\lambda x + S\sin\lambda x + \\ &+ \frac{F}{2\cdot EI\cdot\lambda^3}\cdot\Big[\sinh\lambda(x-x_f) - \sin\lambda(x-x_f)\Big]\cdot H(x,x_f) \end{aligned} \quad (3)$$

where: EI - bending stiffness, A cross-section area, ρ - material density, $\delta(x, x_f)$ - Dirac delta function and $H(x, x_f)$ is Heaviside step function at $x = x_f$.

The relation (3) describes the function of amplitudes of steady-state vibrations caused by force with amplitude F and frequency ω_w, applied in the point $x = x_f$, constants P, Q, R, S can be determine based on the measurement of vibration amplitudes in several points of the beam. The description of the procedure for determining the constants is shortly described in the next section of the paper.

After estimating the integration constants, sought values of support elasticity coefficients can be calculated from the boundary conditions, and for position $x = 0$:

$$EI\cdot X'''(0) = -k_{T0}X(0) \qquad -EI\cdot X''(0) = -k_{R0}X'(0) \quad (4)$$

hence, the lateral and rotational elastic coefficient can be obtained from relations:

$$k_{T0} = EI\cdot\lambda^3\cdot\frac{S-Q}{P+R} \qquad k_{R0} = EI\cdot\lambda\cdot\frac{P-R}{Q+S} \quad (5)$$

The boundary conditions for position $x = l$ are described by:

$$EI\cdot X'''(l) = k_{Tl}\cdot X(l) \qquad EI\cdot X''(l) = -k_{Rl}\cdot X'(l) \quad (6)$$

hence, the lateral elastic coefficient is:

$$k_{Tl} = EI\lambda^3\frac{P\sinh\lambda l + Q\cosh\lambda l - R\sin\lambda l - S\cos\lambda l - f_1}{P\cosh\lambda l + Q\sinh\lambda l + R\cos\lambda l + S\sin\lambda l - f_2} \quad (7)$$

where:

$$\begin{aligned} f_1 &= \frac{F}{2EI\lambda^3}[\cosh\lambda(l-x_f) + \cos\lambda(l-x_f)] \\ f_2 &= \frac{F}{2EI\lambda^3}[\sinh\lambda(l-x_f) - \sin\lambda(l-x_f)] \end{aligned}$$

and the rotational elastic coefficient:

$$k_{Rl} = -EI\lambda\frac{P\cosh\lambda l + Q\sinh\lambda l - R\cos\lambda l - S\sin\lambda l - f_3}{P\sinh\lambda l + Q\cosh\lambda l - R\sin\lambda l + S\cos\lambda l - f_4} \quad (8)$$

where:

$$\begin{aligned} f_3 &= \frac{F}{2EI\lambda^2}[\sinh\lambda(l-x_f) + \sin\lambda(l-x_f)] \\ f_4 &= \frac{F}{2EI\lambda^2}[\cosh\lambda(l-x_f) - \cos\lambda(l-x_f)] \end{aligned}$$

The model has been developed with the assumption that the values of support elasticities are constant, i.e. they do not depend on the amplitude or vibration frequency.

THE METHOD OF DETERMINING THE INTEGRATION CONSTANTS

The simplest method of finding the constants P, Q, R, S is to measure the vibration amplitudes caused by the force of known amplitude and frequency in four points. In this case four constants can be determine from four equations in the form (3) describing the vibration amplitudes in four measuring points.

In many cases of diagnostics or identification, it is not possible to obtain full information on the analysed object. In the case analysed here the ,,incomplete information" is to be understood that the measurement of the vibration amplitudes was taken only in three measuring points.

Assuming that the measuring points are the points with coordinates $x = a$, $x = b$ and $x = c$, (values of the forced vibration amplitudes in these points are marked respectively as $X(a)$, $X(b)$, $X(c)$) we obtain three algebraic equations in the form (3), which can be written in the matrix form: $\mathbf{M} \cdot \mathbf{C} = \mathbf{B}$
hence:

$$\begin{bmatrix} \cosh\lambda a & \sinh\lambda a & \cos\lambda a & \sin\lambda a \\ \cosh\lambda b & \sinh\lambda b & \cos\lambda b & \sin\lambda b \\ \cosh\lambda c & \sinh\lambda c & \cos\lambda c & \sin\lambda c \end{bmatrix} \begin{bmatrix} P \\ Q \\ R \\ S \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{9}$$

where:

$$b_1 = X(a) - \frac{F}{2EI\lambda^3}[\sinh\lambda(a - x_f) - \sin\lambda(a - x_f)] \cdot H(a, x_f)$$

$$b_2 = X(b) - \frac{F}{2EI\lambda^3}[\sinh\lambda(b - x_f) - \sin\lambda(b - x_f)] \cdot H(b, x_f)$$

$$b_3 = X(c) - \frac{F}{2EI\lambda^3}[\sinh\lambda(c - x_f) - \sin\lambda(c - x_f)] \cdot H(c, x_f)$$

The equation (9) is a matrix equation, for which the solution can be obtained by decomposing the main matrix according to Singular Value Decomposition algorithm [3]:

$$\mathbf{M} = \mathbf{U} \cdot \mathbf{W} \cdot \mathbf{V}^T$$

where:

U - a square matrix of order 3 x 3 having 3 orthogonal columns such, that: $\mathbf{U}^T$ **U** = 1,

W - pseudo-diagonal matrix of order 3 x 4 having non-negative singular values on its diagonal,

V - a square matrix of order 4 x 4 having 4 orthogonal columns, such, that: $\mathbf{V}^T$ $\mathbf{V}$ = 1,

So sought constant vector can be determined based on matrix obtained as result of decomposition:

$$\mathbf{C} = \mathbf{V} \cdot \mathbf{W}^{-1} \cdot \mathbf{U}^T \cdot \mathbf{B}$$

After computing the integration constants P, Q, R, S, the sought values of the support elasticities can be calculated from the relation (5), (7) and (8).

COMPARATIVE CRITERIA OF THE MODELS

Verification of the model created based on the data obtained from the experiment is one of the main problems of the identification. The model obtained from identification of the system with ,,incomplete information" (under-determined problem) is an approximation of the real system. The verification stage of identification involves checking of the obtained approximation for sufficiency for the objective, for which the model was created.

The identification objective adopted in the paper is determining the amplitude vectors of steady state vibrations caused by force of any frequency below the second eigenfrequency.

The first basic criterion of comparison of the models of mechanical construction is comparison of their natural frequency. Due to limitations of excitation frequency, the values of two first free vibration frequencies will be compared here.

The second used criterion of analysing the correlation between the models is visual comparison of amplitude vectors of vibrations caused by forces of different excitation frequencies. Mathematical notation of this type of comparison (described and used in modal analysis for finding the correlation between the eigenvectors) is done using MAC (Modal Assurance Criterion) [1] coefficient:

$$MAC(x, y) = \frac{\left|\mathbf{x}^{*T}\mathbf{W_g}\mathbf{y}\right|^2}{(\mathbf{y}^{*T}\mathbf{W_g}\mathbf{y}) \cdot (\mathbf{x}^{*T}\mathbf{W_g}\mathbf{x})}$$

where: $\mathbf{x}^*, \mathbf{x}$ and $\mathbf{y}^*, \mathbf{y}$ are two vectors of the forced vibration amplitudes obtained from the analytical and experimental model; $\mathbf{W_g}$ is a weight matrix indicating, which coordinates of the vector are the most important during comparison. The analysis was performed with the assumption that matrix $\mathbf{W_g}$ is a unit matrix.

NUMERICAL EXAMPLES

The subject of the analysis is elastically supported beam shown in Fig.1 with the following material data: Young modulus $E = 2.1 \cdot 10^{11}$Pa; material density $\rho = 7860$kg/m^3 and geometric data: cross-section bxh=0.03x0.03m; beam length $l = 1.3$m.

The ,,experimental" data required for identification and verification come from the vibration analysis using the finite element method. For this purpose amplitudes were computed for vibrations excited by force of amplitude $F = 100$N applied to the beam in point with a coordinate $x_f = 0.9$m and frequency $\omega = 2\pi f$ (for different frequencies f).

The support elasticity constants were determine based on the measurement (obtained from FEM analysis) of the vibration amplitudes of in three points, whereas two of them are located at the beam ends ($a = 0$, $c = l$). After determining the elasticity coefficients, using decomposition of the main matrix according to Singular Value Decomposition algorithm, the analytical model was verified against the criteria specified above.

Free vibration frequencies were compared by determining the deviation defined by the following formula:

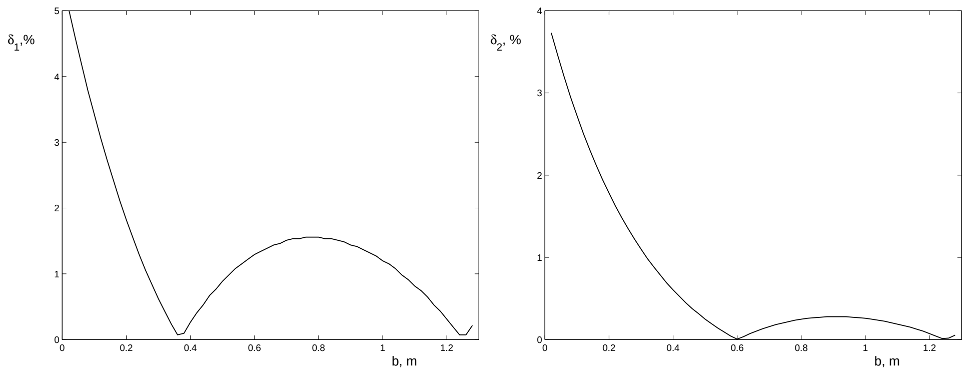
$$\delta_i = \frac{|\omega_{ie} - \omega_{ia}|}{\omega_{ie}} \cdot 100\% \qquad\qquad i = 1,2 \tag{10}$$

where:

ω_{ie} - i-th natural frequency obtained from the experimental model,

ω_{ia} - i-th natural frequency obtained from the analytical model,

Fig. 2 shows the defined above deviation of determining the first and second natural frequency as a function of location of the third measurement point (other two points at the beam ends) for the excitation frequency $\omega = 2\pi \cdot 25 = 157.1$ rad/sec.



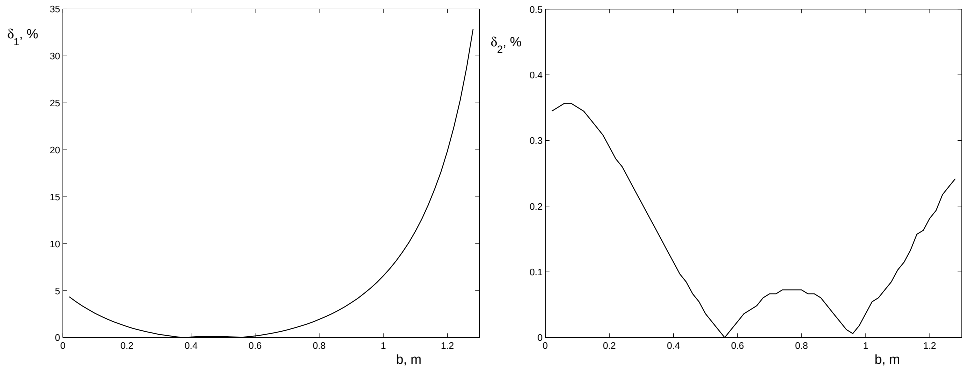
(a) the first natural frequency (b) the second natural frequency

Figure 2: Deviation of determining of the first two natural frequencies as a function of location of the third sensor.

Natural frequencies from FEM analysis: $\omega_1 = 262.1$rad/sec and $\omega_2 = 1039.4$rad/sec.

Fig. 3 shows the defined above deviation of determining the first and second natural frequency as a function of location of the third measurement point for the excitation frequency $\omega = 2\pi \cdot 50 = 314.2$ rad/sec (higher than the first eigenfrequency).

According to the analysis of the results shown in Figs.2 and 3 smaller identification errors are made when system is loaded by a frequency lower than the first frequency of system



(a) the first natural frequency (b) the second natural frequency

Figure 3: Deviation of determining of the first two natural frequencies as a function of location of the third sensor.

free vibration. In such case location of the central sensor (other two are located on the beam ends) has no significant effect on uncertainty of the determining of two first eigenfrequencies (deviations δ_1 and δ_2 below 5%).

In the case of identification with excitation by force of a frequency higher than first natural frequency it is essential (due to identification error) to find ,,appropriate" position of the measuring element (frequency determining deviation varies from 0 to 33%).

Later on we will determine the correlation coefficients for the vectors of amplitudes of forced vibration obtained from the experiment $\mathbf{X}_e$ and analytical model $\mathbf{X}_a$:

$$MAC(\mathbf{X}_a, \mathbf{X}_e) = \frac{\left|\mathbf{X}_a^T \cdot \mathbf{X}_e\right|^2}{(\mathbf{X}_a^T \cdot \mathbf{X}_a) \cdot (\mathbf{X}_e^T \cdot \mathbf{X}_e)}$$

Table 1 summarizes the MAC coefficients computed by comparison of forced vibration vectors, obtained for 7 different frequencies of exciting force. The boundary conditions necessary to calculated the vectors of vibrations from the analytical model are obtained from the identification measurements with a excitation frequency $\omega = 2\pi \cdot 25 = 157.1$rad/sec and $\omega = 2\pi \cdot 50 = 314.2$rad/sec.

Table 1: Correlation coefficients MAC for vibration amplitude vectors obtained from the analytical and experimental model

Identification with $\omega = 2\pi \cdot f$	Verification for $\omega = 2\pi \cdot f$						
	$f = 10$	$f = 25$	$f = 50$	$f = 75$	$f = 100$	$f = 125$	$f = 150$
$f = 25$Hz	0.9983	0.9982	0.9975	0.9989	0.9930	0.9965	0.9953
$f = 50$Hz	0.9998	0.9997	0.9976	0.9992	0.9992	0.9907	0.9811

Identification and verification measurements were performed for the case, where the central measuring point was in the beam center ($b = l/2$).

Verification results for the analytical model indicate correct identification of the boundary conditions of the beam, at least in the assumed band of excitation frequency.

CONCLUSIONS

The purpose of the study was demonstration of possibility of beam boundary conditions identification in under-determined problem. In the analysed case the problem is to be understood as determining four constants necessary for description of the functions of forced vibration amplitudes, from three equations. These equations described amplitudes of vibrations in points, where measuring elements (sensors) were located.

Verification of the obtained mathematical model (elastically supported Bernoulli-Euler's beam) was done by: comparing natural frequencies obtained from the analytical model and numerical experiment, and analysing the correlation of forced vibration amplitude vectors for different excitation frequencies.

The deviation of determining the first and second free vibration frequency was computed as a function of location of one measuring element. Identification of boundary conditions was done for two frequencies of exciting force: below and above the first eigenfrequency.

According to the results of the analysis shown on Figs.2 and 3, smaller identification errors are made when system is loaded by a frequency lower than the first eigenfrequency. However, it is essential to find such position of the measuring element, which will ensure minimal identification error.

Another criterion used in the analysis for studying the correlation between the models (described and used in modal analysis for finding the correlation between eigenvectors) is MAC (Modal Assurance Criterion) coefficient. MAC coefficients for the vectors of forced vibration obtained from the experiment and analytical model are summarized in table 1. All MAC coefficients are greater than 0.8, above which, good coincidence of vectors is assumed.

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