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MODERATELY LARGE FLEXURAL VIBRATIONS OF LAYERED PANELS WITH INITIAL IMPERFECTIONS

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Abstract

Nonlinear flexural vibrations of doubly curved layered panels composed of three thick and with respect to the middle surface layers symmetrically arranged layers are analyzed. The panels are of polygonal planform and hard hinged simply supported with the edges fully restraint against displacements. The governing equations are derived by application of a first order shear deformation theory, and nonlinear structural behavior is considered by a modified Berger theory. Numerical results of rectangular panels with initial imperfections in nonlinear steady state vibration exhibit layerwise different cross-sectional rotations as main dynamic effect.

BASIC EQUATIONS

A panel of three thick isotropic and homogeneous layers in perfect bond is considered. Thickness and linear elastic properties of the outer layers are identical, however the inner layer may have a considerably different elastic modulus. All layers exhibit the same Poisson's ratio ν. A small initial geometric imperfection $\hat{w}$ of the middle surface is referred to the coordinates *x* and *y* in the projected (*xy*-) plane of the middle surface. The origin (z = 0) of the perpendicular coordinate *z* is the curved middle surface of the panel (and not the *xy*-plane) [1]. The boundaries of the panel with polygonal planform are modeled as hard hinged supported edges with the displacements perpendicular to the edge face fully restrained. All edges are straight, i.e. the imperfection $\hat{w}$ is zero at the boundaries, and thus the middle surface of the doubly curved panel and of the *xy*-plane coincide at the edges. The mechanical description of the panel is based on a layerwise first order shear deformation theory, and thus, the displacement field of the *i*th layer may be expressed as [2],

$$w_i = w\,,\ {}_iu_j = {}_iu_j^{(0)} + z\ {}_i\psi_j\,,\quad i = 1,2,3,\ j = x,y \tag{1}$$

w denotes the lateral displacement in z-direction common to all layer surfaces, which is superposed to the initial imperfection $\hat{w}$. ${}_iu_x$, ${}_iu_y$ represent in-plane displacements in the ith layer in x and y direction, respectively, at distance z from the middle surface. ${}_iu_x^{(0)}$, ${}_iu_y^{(0)}$ denote in-plane displacements at z = 0, and ${}_i\psi_x$, ${}_i\psi_y$ are layerwise cross-sectional rotations, i = 1, 2, 3. Note that index i = 2 refers to quantities of the inner layer, and i = 1 and 3 belong to the upper and lower outer layer, respectively. Assuming perfect bond between the layers the in-plane displacements ${}_iu_x^{(0)}$, ${}_iu_y^{(0)}$ of the faces (i = 1, 3) are expressed in terms of the in-plane displacements of the middle surface ${}_2u_x^{(0)}$, ${}_2u_y^{(0)}$ and the cross-sectional rotations [3, 4]:

$${}_iu_j^{(0)} = {}_2u_j^{(0)} + z_i\left({}_2\psi_j - {}_i\psi_j\right)\,,\quad i = 1,3,\quad j = x,y \tag{2}$$

where $z_1 = -h_2/2$, $z_3 = h_2/2$ denote perpendicular distances from the middle surface to the upper and lower interface, respectively.

The transverse displacement component w is assumed not to be small compared to the panel thickness h, and thus, the interaction between the membrane stresses and the curvatures must be considered. This interaction results in the stretching of the middle surface and subsequently to nonlinear terms in the strain-displacement relations [1],

$${}_i\varepsilon_j = {}_iu_{j,j} + \left(w_{,j}\right)^2/2 + w_{,j}\,\hat{w}_{,j}\,,\quad {}_i\gamma_{xy} = {}_iu_{x,y} + {}_iu_{y,x} + w_{,x}\,w_{,y}\,, \tag{3.1}$$

$${}_i\gamma_z = {}_iu_{j,z} + w_{,j}\,,\quad i = 1,2,3,\ j = x,y \tag{3.2}$$

${}_i\varepsilon_x$, ${}_i\varepsilon_y$ are the strains in x- and y-direction, respectively, and ${}_i\gamma_{xy}$ denotes the in-plane shear strain.

The constitutive relations, however, are linear. For an isotropic, elastic material the stress components σ_x, σ_y, τ_{xy} are related to the strains by means of Hooke's law, see e.g. [5]. The normal stress component σ_z is assumed to be negligible and consequently dropped.

Transverse shear stress components τ_{xz}, τ_{yz} are specified to be continuous across the interfaces according to Hooke's law [3, 6, 7],

$${}_i\tau_{jz} = G_i\left({}_i\psi_j + w_{,j}\right) = {}_{i+1}\tau_{jz} = G_{i+1}\left({}_{i+1}\psi_j + w_{,j}\right)\,,\quad i = 1,2,\quad j = x,y \tag{4}$$

G_i is the shear modulus of the isotropic ith layer. In analogy to the Mindlin theory for homogeneous panels Eqs (4) exhibit the simplified assumption that the shear stress is uniformly distributed throughout the layers. From this relation and in combination with $G_1 = G_3$ it follows that the cross-sectional rotations of both faces are identical: ${}_1\psi_x = {}_3\psi_x$ and ${}_1\psi_y = {}_3\psi_y$.

Layerwise stress resultants are determined by integration of the stress components with respect to the thickness of the layers. Utilizing Eqs (4) the cross-sectional rotations of the faces are eliminated, and the layerwise resultants can be expressed in terms of the lateral deflection *w*, the cross-sectional rotations ${}_2\psi_x$, ${}_2\psi_y$ of the core, the middle surface strains and their derivatives, see [7].

The in-plane displacements of the middle surface at the edges Γ are fully restrained, i.e. ${}_2u_x^{(0)}\big|_\Gamma = 0$, ${}_2u_y^{(0)}\big|_\Gamma = 0$, and thus, moderately large lateral displacements may be considered simplified by means of Berger's approximation [8]. This approximation is based on the assumption that the elastic energy given by the second invariant of the middle surface strain tensor may be disregarded as compared to the square of the first invariant without substantially affecting the response. In Berger's approach the influence of the in-plane force resultants is characterized by a time-variant isotropic force *n*, which is a constant throughout the panel domain Ω. Following a procedure employed by Wah [9] and Irschik [10] *n* may be related to the lateral deflection *w* and the initial imperfection $\hat{w}$ by the averaging integral

$$n = \frac{D}{\Omega}\int_\Omega \left[\frac{1}{2}\left(w_{,x}^2 + w_{,y}^2\right) + w_{,x}\,\hat{w}_{,x} + w_{,y}\,\hat{w}_{,y}\right] d\Omega\ ,\quad D = \frac{2}{1-\nu}\sum_{i=1}^{3} G_i\, h_i \tag{5}$$

EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

The equations of motion are derived considering the free-body diagram of an infinitesimal panel element, loaded by a lateral forcing function $p(x, y; t)$. Thereby, in a common approximation, both, the longitudinal as well as the rotatory inertia are neglected, ${}_i\ddot{u}_j^{(0)}$, ${}_i\ddot{\psi}_j$, thus, limiting the analysis to the lower frequency band of structural dynamics. Conservation of angular momentum with respect to the *x*- and *y*-axes and conservation of momentum in *x*-, *y*-, and *z*-direction render after some algebra the following equations of motion of the nonlinear panel problem [7]:

$$-S_e\left(\Delta w + {}_e\psi_{x,x} + {}_e\psi_{y,y}\right) - n\left(\Delta w + \Delta\hat{w}\right) + \mu\,\ddot{w} = p \tag{6.1}$$

$$-K\left({}_e\psi_{j,jj} + \frac{1-\nu}{2}\,{}_e\psi_{j,kk} + \frac{1+\nu}{2}\,{}_e\psi_{k,jk}\right) - S_e\left(w_{,j} + {}_e\psi_j\right) = 0\ , \tag{6.2}$$

$$j = x,\ k = y \text{ and } k = x,\ j = y$$

where ${}_e\psi_x$ and ${}_e\psi_y$ denote effective cross-sectional rotations [3, 4],

$${}_e\psi_j = \frac{\delta}{S_e}\,{}_2\psi_j + \left(\frac{\delta}{S_e} - 1\right) w_{,j}\ ,\quad j = x, y \tag{7}$$

Expressions (6) may be understood as the equations of motion of an imperfect homogeneous shear-deformable panel with mass per unit area μ, effective shear stiffness S_e, and flexural stiffness *K*. The effective panel properties are given by

$$\mu = 2\rho_1 h_1 + \rho_2 h_2\ ,\quad K = \frac{2}{1-\nu}\left(2C_1 + C_2\right)\ ,\quad S_e = \frac{\delta}{\gamma} K \tag{8.1}$$

$$\delta = \kappa^2 G_2 h \ , \ \ \gamma = \frac{2}{1-\nu}\left[B_1 h_2 \left(\frac{G_2}{G_1} - 1 \right) + 2 C_1 \frac{G_2}{G_1} + C_2 \right] \tag{8.2}$$

$$B_1 = \frac{G_1}{8}\left(h_2^2 - h^2\right) \ , \ \ C_1 = \frac{G_1}{24}\left(h^3 - h_2^3\right) \ , \ \ C_2 = \frac{G_2}{12} h_2^3 \tag{8.3}$$

In S_e a shear correction factor κ^2 is employed, since the transverse shear stresses are assumed to be constant through the thickness. This is the same concept used in the Mindlin-Reissner theories for thick plates. In (8.1) ρ_1, ρ_2 denote mass densities of the faces and the core, respectively.

Considering only polygonal contours Γ (i.e. straight edges) the boundary conditions of a composite shear-deformable panel with hard hinged supports can be modeled in the form [7]

$$\Gamma: \ \ w = 0 \ , \ \ {}_e\psi_s = 0 \ , \ \ {}_e\psi_{n,n} = 0 \tag{9}$$

where *n*, *s* are local Cartesian coordinates at boundary Γ with normal *n* pointing outwards.

Eqs (6) and (9) are solved for the kinematic variables w, ${}_e\psi_x$, ${}_e\psi_y$. In a subsequent step the individual cross-sectional rotations of the inner layer ${}_2\psi_x$, ${}_2\psi_y$ and of the outer layers ${}_1\psi_x$, ${}_1\psi_y$ are derived by decomposition of Eqs (7) and (4),

$${}_2\psi_j = {}_e\psi_j \frac{S_e}{\delta} - \left(1 - \frac{S_e}{\delta}\right) w_{,j} \ , \ \ {}_i\psi_j = \frac{G_2}{G_1}\left({}_2\psi_j + w_{,j}\right) - w_{,j} \ , \ i = 1,3, \ \ j = x, y \tag{10}$$

NONLINEAR HARMONIC STEADY-STATE RESPONSE

In the following the dynamic steady-state response of rectangular imperfect layered panels to time-harmonic excitation is studied. The panels of length *a*, width *b* and thickness *h* are composed of three layers with layer to overall thickness ratios of $h_{1(3)} / h$ =1/4 and h_2 / h =1/2. The overall dimension is characterized by the aspect ratio a / b =3/4 and by the thickness to length ratios h / a =0.08 and 0.05, respectively. The mechanical properties of the outer layers and the inner layer are specified through the ratio $G_{1(3)} / G_2$ =20. Poison's ratio ν for all layers is 0.3, and a shear coefficient κ^2 of 1 is considered. The imperfection follows a sine half-wave according to $\hat{w} = \hat{w}_0 \sin\left(\pi y / b\right) \sin\left(\pi y / b\right)$, where $\hat{w}_0$ denotes the maximum initial displacement in the center of the panel. Note that the *x* and *y* coordinates point in the direction of length *a* and *b*, respectively, and their origin is in a corner of the panel.

For the solution of the actual boundary value problem, Eqs (6) and (9), the kinematic variables are expanded into the ortho-normalized set of mode shapes of the corresponding linearized panel [7]. Note that the mode shapes are not affected by the imperfection, which is proportional to the fundamental mode shape [7, 11]. The procedure of modal analysis renders a coupled set of nonlinear ordinary differential equations for modal coordinates, where structural damping is introduced via modal

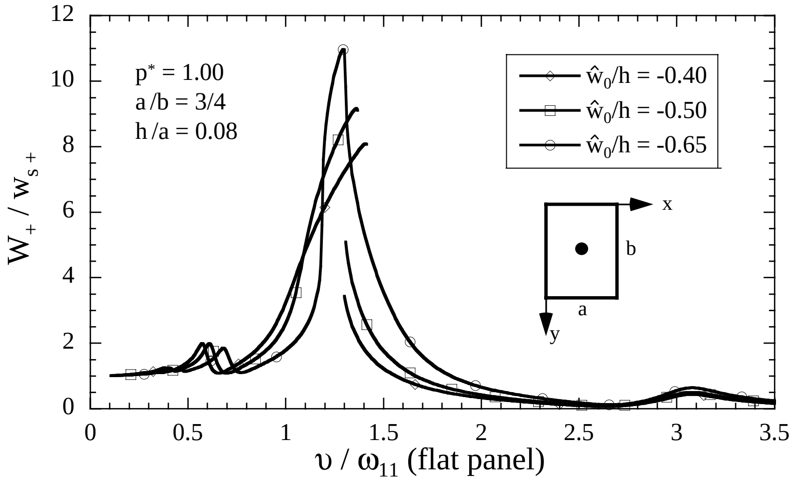

Figure 1 – Amplitude functions of the central lateral deflection for different magnitudes of imperfections

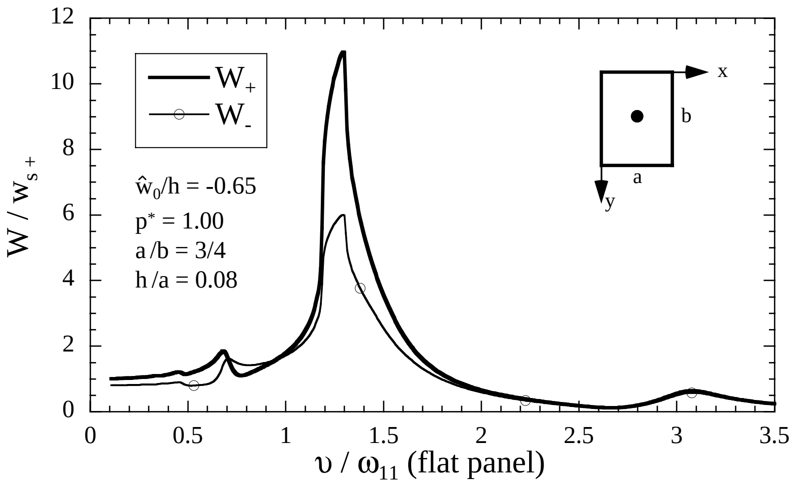

Figure 2 - Amplitude functions of the central lateral deflection in and towards the direction of the center of imperfection curvature

damping coefficients ζ. For further details see [7]. In this study ζ is selected to be 5% for all modes. In particular, frequency response functions due to a uniformly distributed lateral load $p(x,y;t) = p_0 \sin \upsilon t$ are derived by sweeping the excitation frequency υ. The load amplitude p_0 is presented in non-dimensional form, $p^* = p_0 a^3 / K$. At time t = 0 the load is subjected to the panel, and the nonlinear modal equations are solved performing a time-history analysis. Thereby, the infinite modal series are approximated by considering the first 4 symmetric modes. After decay of the transient response the maximum steady-state response is recorded.

Figure 1 shows non-dimensional amplitude functions of the lateral deflection at the center of the thick panel (i.e. h/a=0.08) with three different imperfection amplitudes, which are characterized by the ratio $\hat{w}_0 / h$. The non-dimensional load amplitude p^* is 1.00. The amplitude functions W_+ are normalized by means of the corresponding static central deflection w_{s+} due to the static external pressure $p = p_0$.; and the excitation frequency υ is related to the fundamental frequency ω_{11} of the corresponding linear flat panel ($\hat{w}_0 / h$ =0) with identical shape. Thereby, W_+ is the inward displacement amplitude (in direction of the center of curvature), which is different from the outward amplitude W_- , see Figure 2 for a panel with $\hat{w}_0 / h$ =-0.65. From Figure 1 it can be seen that with increasing imperfection amplitude the pronounced nonlinear hardening response behavior gradually passes over to softening behavior. The bending deformation of the resonance curves leads for $\hat{w}_0 / h$ =-040 and -0.50 to multivalued amplitudes, and the entire solution splits into stable and unstable branches. However, in Figure 1 only the stable portions of the response are displayed. For all considered panels the influence of subharmonic resonance becomes visible by additional peaks at about half of the linearized primary resonance frequency. For an imperfect panel with $\hat{w}_0 / h$ =-0.65 in Figure 3 response amplitudes W_+ for different loads p^* are displayed, and it can be seen that the type of nonlinearity (hardening or softening) strongly depends on the load magnitude p^* .

Figure 4 presents inward amplitude response functions of the individual cross-sectional rotations ${}_1\psi_x$ and ${}_2\psi_x$, the effective cross-sectional rotation ${}_e\psi_x$, and the derivative of the displacement with respect to *x*, $w_{,x}$, at point (*x*/*y* = 0/0.5 *b*) for the same panel structure. All individual rotation amplitudes are normalized by means of the corresponding effective cross-sectional rotation ${}_e\psi^s_{x+}$ due to the static external pressure $p = p_0$. A non-dimensional load amplitude of p^* =1.00 is considered. It can be seen that the cross-sectional rotations of the outer layers and of the inner layer do not coincide. In the primary response domain the amplitudes ${}_1\Psi_{x+}$ of the outer layers are larger than that of the inner layer (${}_2\Psi_{x+}$), in the vicinity of the second excited mode ${}_2\Psi_{x+}$ exceeds ${}_1\Psi_{x+}$. The amplitude response ${}_1\Psi_{x+}$ is slightly overestimated by $W_{,x}$ at the primary resonance frequency, otherwise the corresponding graphs are identical. Normalized amplitude response functions of the cross-sectional rotations ${}_1\psi_y$, ${}_2\psi_y$, ${}_e\psi_y$, $w_{,y}$ at (*x*/*y* = 0.5 *a* / 0) are depicted in Figure 5.

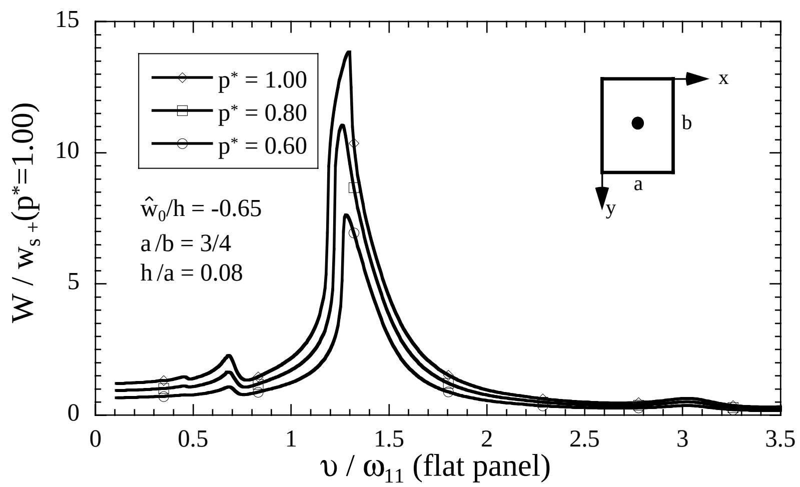

Figure 3 - Amplitude functions of the central lateral deflection for different non-dimensional load amplitudes

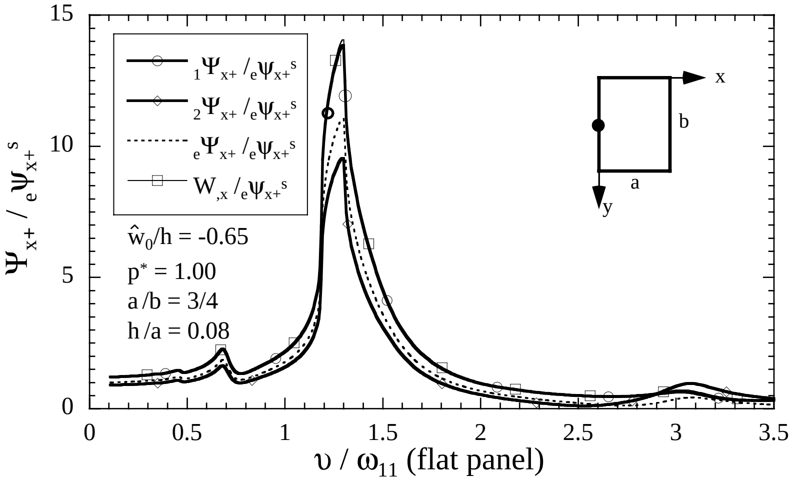

Figure 4 - Amplitude functions of the layerwise and the effective cross-sectional rotations, and of the derivation of the deflection with respect to x at (x/y=0/0.5b)

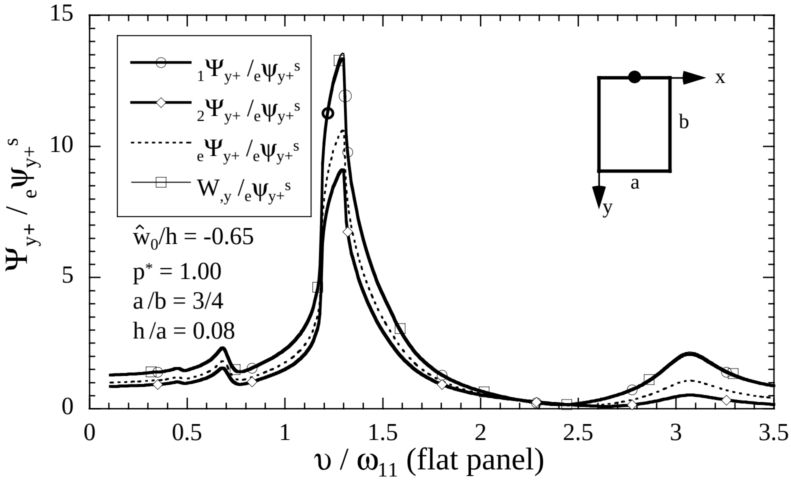

Figure 5 - Amplitude functions of the layerwise and the effective cross-sectional rotations, and of the derivation of the deflection with respect to y at (x/y=0.5a/0)

Subsequently, a thin panel with thickness to length ratio $h/a = 0.05$ and a rise $\hat{w}_0/h$=-0.65 is considered. In Figure 6 the normalized amplitude functions of the cross-sectional rotations $_1\psi_x$, $_2\psi_x$, $_e\psi_x$ and of $w_{,x}$ at point (*x*/*y* = 0/0.5 *b*) are presented for an applied load of p^*=1.00. It can be seen that for this thin panel the cross-sectional rotations of the outer and inner layers are almost identical in the quasi-static frequency range. With increasing excitation frequency a difference between the individual cross-sectional rotations becomes apparent. From this result it can be concluded that the importance of a layerwise description of the displacement field is not only a function of geometry and material parameters but also of the considered frequency range.

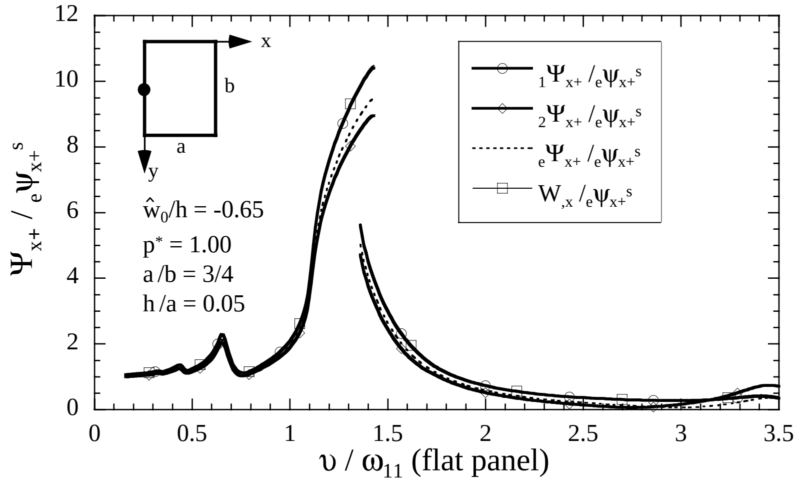

Figure 6 - Amplitude functions of the layerwise and the effective cross-sectional rotations, and of the derivation of the deflection with respect to x at (x/y=0/0.5b); thin panel

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