RESONANT ACOUSTIC MODEL OF CATASTROPHIC TSUNAMI

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Abstract

A quadratic nonlinear ordinary equation for coastal waves is presented. The equation takes into account the nonlinear interaction between dispersion, variable depth and bottom frictional dissipation. According to this equation the amplification and the evolution of the tsunami may be connected with nonlinear resonance of ocean waves near the coast line. A tsunami is considered as a forced solitary-like wave. Analytic solutions which describe data of experiments and numerical calculations are presented. It is found that the nonlinear resonance can explain the catastrophic coastal amplification of tsunami.

INTRODUCTION

The concept of resonance [1-3] is used to explain the coastal evolution of a tsunami. It was shown that the coastal zone forms the resonant band where tsunami speed is very close to or coincides with the wave phase speed. The catastrophic evolution takes place as the tsunami passes this band.

Tsunamis have been studied in many publications and powerful numerical methods developed [5, 7, 10]. For example, Titov *et al*. [10] developed a numerical method which can solve problems connected with the generation, the propagation and the coastal evolution of tsunamis. There are also important analytic investigations of tsunami evolution [6, 8, 9]. As a result, the propagation of tsunami in the deep ocean is understood well. In contrast, the coastal evolution of tsunami is more difficult to simulate since this evolution is governed by many parameters. Which parameters are the most important is difficult to determine. Here we introduce a transresonant parameter R. Linear and nonlinear theories give the same results when the tsunami passes the continental slope and $|R| >> 1$. The catastrophic amplification takes place within the coastal zones where $|R| < 1$.

NONLINEAR RESONANCE OF COAST OCEAN WAVES AND TSUNAMI

As the tsunami model we use a unidirectional travelling wav $f = f(\tau)$, $\tau = t - aC^{-1}(a)$. Here t is time, a is the Lagrangian coordinate and C is the wave velocity. The evolution of this wave near the coast is described by a quadratic nonlinear ordinary equation [1-3]

$$[(C^2 - gh) - 3ghC^{-1} f^{'}] f^{''} = \tfrac{1}{3} h^2 f^{''''} + 2\mu\rho^{-1}h^{-1}C_a f^{''} \\ + g(2hC_a - h_a C) f^{'} + (\mu\rho^{-1}h^{-1} - 0.5hh_a C + \tfrac{2}{3} h^2 C_a) f^{'''} + C^2 X. \quad (1)$$

Here g is the acceleration due to gravity, $h = h(a)$ is the depth, μ is the coefficient of the bottom friction and X the component of the body force. The subscript a indicates derivative with respect of time.

In (1) the linear and nonlinear acoustic terms have been written on the left, and the various second order terms which arise from dispersion, the bottom friction, and the variable depth appear on the right. It is seen that the left hand side terms annihilate each other at the point where

$$C^2 = gh + 3ghC^{-1} f^{'} . \quad (2)$$

This point may be called the nonlinear resonance.

Below we consider particular cases of (1). The tsunami is modelled as a solitary – like wave.

1. *Linear Waves*: For linear free waves eq (1) yields

$$C^2 f^{''} - ghf^{''} = \tfrac{1}{3} h^2 f^{''''} . \quad (3)$$

Let $f^{'} = \mathrm{sech}^2(\omega\tau)$. Then we derive the dispersive relation $C^2 - gh = -\tfrac{2}{3}\omega^2 h^2$.

Let $C = \sqrt{gh} + C_1$, where $\sqrt{gh} >> C_1$. If h is small enough then

$$C_1 = -\tfrac{1}{3}\omega^2 h^2 (gh)^{-0.5} . \quad (4)$$

Thus C_1 takes into account the weak dispersion effect. It is seen that (4) depends on the wavelength and the depth. The larger the wavelength, the weaker the dispersion.

2. *Tide-like wave:* Following [1-3, 5] we assume that the earthquake-induced tsunami may be considered as tide-like wave which is excited by the body force *X*. We shall model the tsunami as a solitary wave. The forcing (incident) wave is written in the form $\int X \mathrm{d}\tau = \varepsilon \mathrm{sech}^2(\omega\tau)$. In this case the linear expression for the water elevation is

$$\eta = -hu_a = hf^{'}(aC^{-1})_a \approx -1.5\varepsilon g^{0.5} h^{-0.5} \omega^{-2} \sec h^2(\omega\tau). \qquad (5)$$

3. *Nonlinear resonance:* Now we rewrite (1) at the vicinity of the nonlinear resonance [3]

$$A(f^{'})^2 + Bf^{'} + \chi f^{''} - \varepsilon \mathrm{sech}^2(\omega\tau) = \overline{C}. \qquad (6)$$

Here $\overline{C}$ is a constant of integration. We assume that $\overline{C}$ = 0. Coefficients *B* and χ are determined as $A = -\omega^2 h^2 (gh)^{-1.5}$, $B = -\frac{2}{3}(gh)^{-1} h^2 \omega^2$ and $\chi = 1/6\, h_a g^{-0.5} h^{0.5}$ $-\mu\rho^{-1} g^{-1} h^{-2}$.

Eq (6) describes the nonlinear evolution of ocean waves near a coast line. According to this equation, near the resonance the linear effect is reduced and may be of second order value. As a result, the influence of nonlinearity, bottom friction, dispersion and depth on the coastal evolution of the waves increases.

4. *Near-resonance solution for weak bottom friction:* Let the effective friction χ be very small. Then eq (6) transforms into an algebraic quadratic equation. Using the solution of this equation we find

$$\eta \approx -hu_a = 0.5hA^{-1}C^{-1}[-B + \sqrt{B^2 + 4\varepsilon A \mathrm{sech}^2(\omega\tau)}] \ . \qquad (7)$$

5. *Near-resonance solution:* Now we consider the full equation (6) which is rewritten in the form

$$(f^{'} + 0.5BA^{-1})^2 - 0.25B^2A^{-2} + \chi A^{-1} f^{''} - A^{-1}\varepsilon \sec h^2 \omega\tau = \overline{C} \ . \qquad (8)$$

Let $0.25B^2 A^{-2} + \overline{C} = 0$. Now following [3] we find that

$$\eta \approx hC^{-1} f' = hC^{-1}\sqrt{A^{-1}\varepsilon}[-2R\pi^{-1} + \tanh\{2\sqrt{A\varepsilon}\omega^{-1}\chi^{-1} \arg\tan[\exp(\omega\tau)]\}\sec h(\omega\tau)], \tag{9}$$

where R is a transresonant parameter: $R = 0.25BA^{-1}\pi / \sqrt{A^{-1}\varepsilon}$. Generally speaking the expressions (7) and (9) are valid only in the vicinity of the resonance.

COAST TRANSRESONANT TSUNAMI EVOLUTION: RESULTS OF TESTS AND CALCULATIONS

The aim of the calculations is to check the theory. Therefore in this section we use the analytic solutions (5), (7) and (9) to simulate Synolakis' experimental data [5, 8, 9] and Smith's numerical data [7].

Since the bottom friction coefficient μ is not a well determined value we consider μ to be an arbitrary value. Similarly, it is known that near a beach the local wave speed is not well determined. Different points on the nonlinear wave, such as points of a wave crest and points of a steep wave front, have different time-varying speeds. At the same time, it is known from Russell's experiments [4], that the speed of a solitary wave depends on the maximum elevation of a wave: $\overline{\eta} = \overline{\eta}(h)$. Therefore we assume in (5), (7) and (9) that

$$\omega\tau = \omega(t - aC^{-1}) = \omega\{t - a[(gh + g\overline{\eta})^{0.5} + C_1]^{-1}\}. \tag{10}$$

The value of $\overline{\eta}$ is calculated for the maximal amplitude of the forced wave according to (7): $\overline{\eta} = 0.5hA^{-1}C^{-1}(-B + \sqrt{B^2 + 4\varepsilon A})$. Values ε and ω in the solutions are chosen so that the calculated results for points that are located far from the coast line match the experimental and numerical data. The solutions are then used to calculate the wave profiles for other points and time.

1. *Synolakis' data [5, 8, 9]:* These data are simulated as a nonlinear resonance phenomenon. It is suggested that $\mu\rho^{-1} = 0.0005$ m^{2}sec^{-2} and the bottom slope is 1:19.85. First we simulate data presented in Figs 8 (*a*-*c*) from [5] assuming that $\varepsilon = -0.1125$ m/sec, ω=1.4 sec^{-1}. Results of the calculations are presented in Fig.1.

The profiles, that are drawn as thin lines, are calculated according to the equation: $Bf' = \varepsilon \sec h^2(\omega\tau)$. Profiles, that are drawn as thick lines, are calculated according to solution (7). Here, and below, lines *a* and *b* show the dependence of coefficients (-*A*) and (-*B*) in (7) on *h*. The line 'FRICTION' shows, here and below, the variation of the value $1000\,\mu\rho^{-1}h^{-2.5}g^{-0.5}h_a$, which is proportional to the bottom friction, with depth. One can see that the amplitude of the nonlinear waves is close to the observed amplitude. The nonlinear profiles also describe the steepness of the wave front near the cost line.

Data from [8, 9] are now simulated. Results are presented in Figs 2 and 3.

Profiles a , b and c are calculated according to equation $Bf' = \varepsilon \sec h^2(\omega\tau)$ for t = 2, 4, 8 sec. Profiles 1, 2 , 3 and 4 are calculated according to the nonlinear solution (7) for t = 2, 8, 10 and 20 sec.

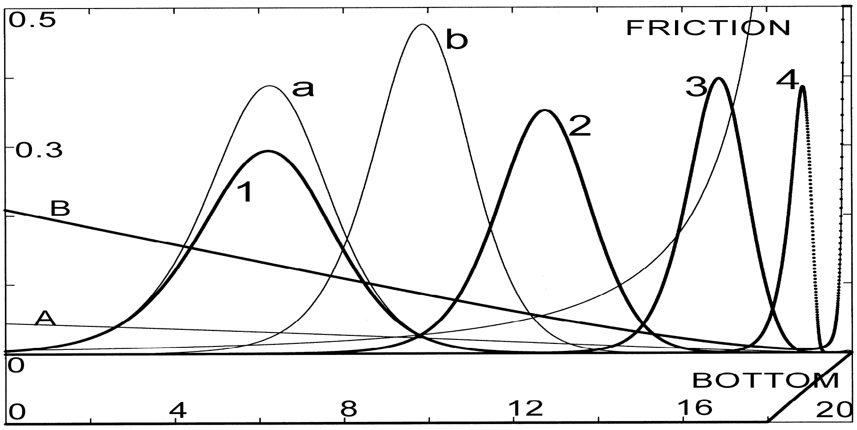

Figure 1 - Profiles a and b are calculated for t = 2, 3.5 sec. Profiles 1, 2 , 3 and 4 are calculated for t = 2, 5, 8 and 11 sec.

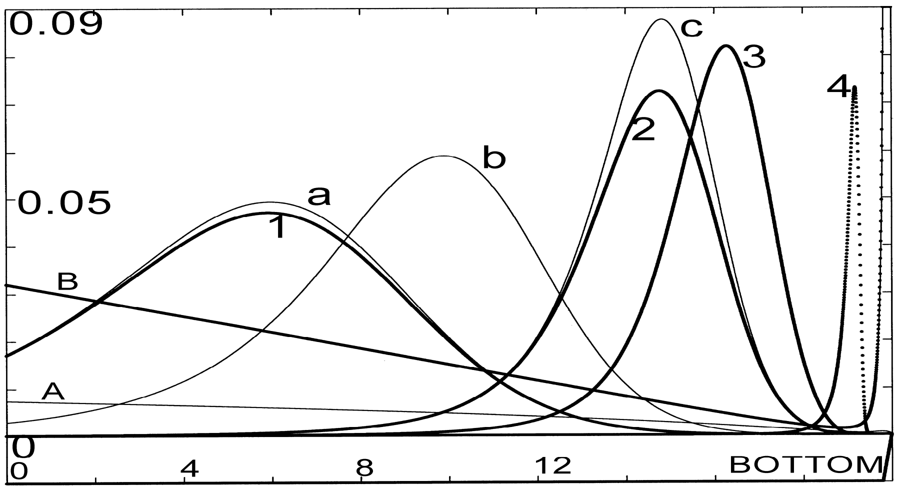

Figure 2 - Simulation of experimental data from Synolakis ([5], Figure 8 (a-c)): $\varepsilon = 0.00215$ *m/sec,* ω *=0.5 sec*$^{-1}$*.*

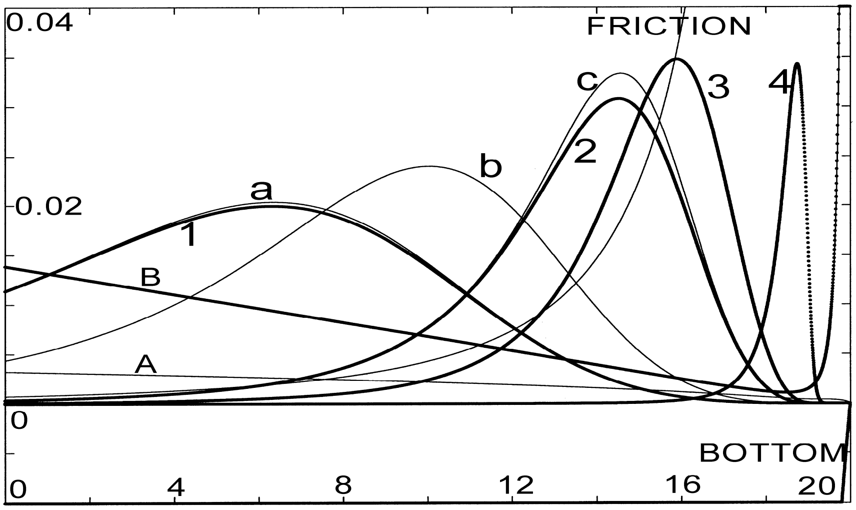

Figure 3 - Simulation of experimental data from Synolakis ([5], Figure 6 (a-d)):
$\varepsilon = 0.00215$ *m/sec,* $\omega = 0.5$ *sec*$^{-1}$.

One can see that the nonlinear solution (7) describes the coastal evolution of a solitary wave which was observed in the experiments. Thus the coastal evolution of the solitary-type wave may be connected with the nonlinear resonance.

2. *Smith's numerical data[7]:* Using (5), (7) and (9) we simulate numerical data presented in Figures 7 and 8 in [7]. The results are shown in Figures 4 and 5. The lines marked by *a* are drawn according to the linear solution (5), the thin lines are drawn according to analytic nonlinear solution (9) and the thick lines (points) are calculated with the help of solution (7). The line *R*/2 describes the variation of the transresonant parameter.

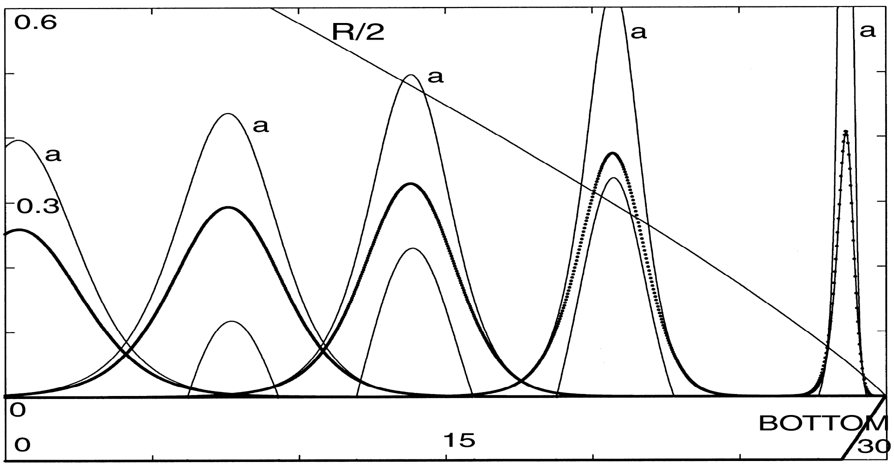

Figure 4 - Nonlinear resonance. The simulation of Figure 7 [9].

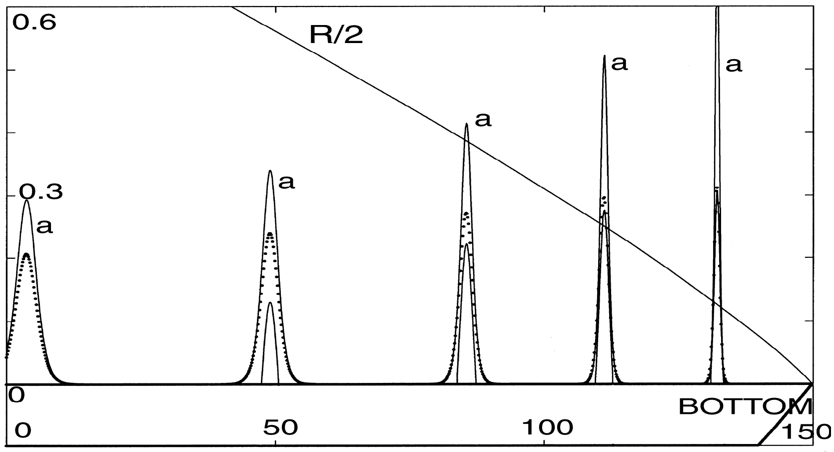

Figure 5 - Nonlinear resonance. The simulation of Figure 8 [9].

Comparing Smith's numerical data [9] and Figures 4 and 5 one can see that the linear solution approximately describes the numerical data only if R >1. If R <1 then the amplitude of the linear wave can be much larger the numerical data. However if R <1 then the nonlinear solutions are applicable. In particular, analytic solution (9) is approximately valid if R <0.6, while solution (7) is valid within and near the resonant band (following [1-3] we determine this band as a field where $|R| \le 1$).

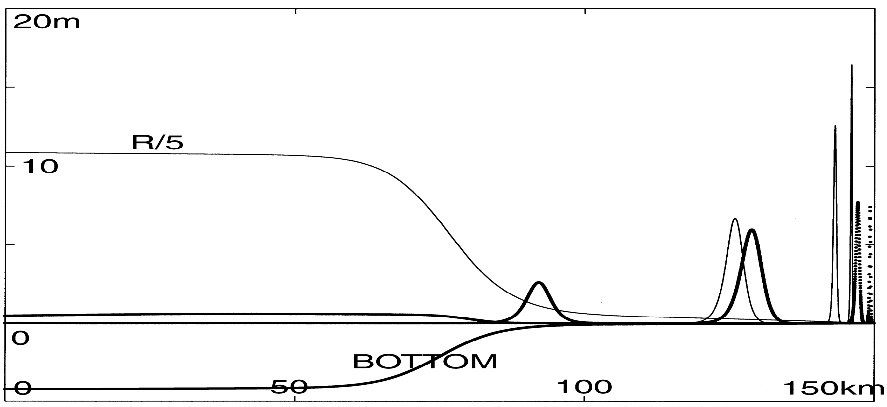

Figure 6 – The profiles are calculated for t =200, 2000, 7000, 15000, 20000 sec.

The interaction of the tsunami ($\varepsilon = -0.00005$ m/sec; $\omega = 0.0025$ 1/sec) and the continental slope is shown in Figures 6 and 7. Results of the linear theory (5) (thin lines) and the nonlinear theory (7) (thick line and points) show that the theories give same results when the tsunami passes the continental slope and $|R| >> 1$. The difference of the results and the catastrophic amplification appears only within the coastal zone where $|R| < 1$.

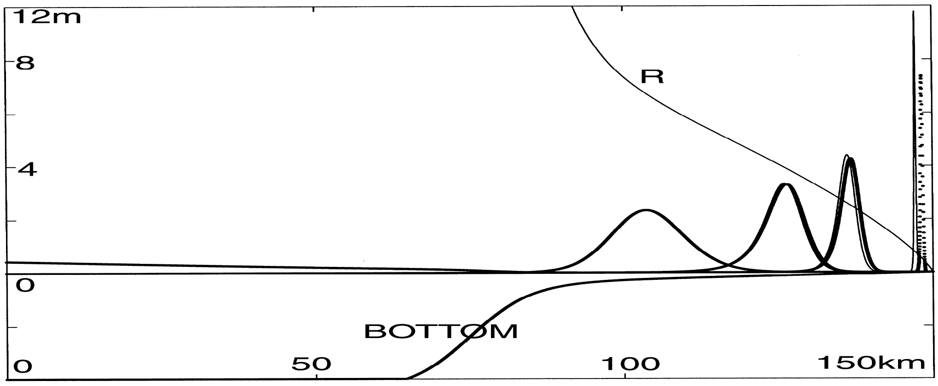

Figure 7 – The profiles are calculated for t =200, 2000, 7000, 15000, 20000 sec.

CONCLUSIONS

Equation (1) approximates the evolution of shallow ocean waves. It was found that the coastal evolution of tsunamis is located within resonant band and connected with the nonlinear resonance. According to the calculations the nonlinear theory is applicable near the coast line, while the linear approximation is valid far from the coast. This agrees with Synolakis' conclusion [8, 9].

The solutions and the results of the calculations clearly show that the coastal evolution of a tsunami is a result of nonlinear resonance. During this transresonant evolution the solitary-like wave can be transformed into the shock-like wave (the wall of the water).

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