ICSV13 - Vienna
The Thirteenth International Congress on Sound and Vibration
Vienna, Austria, July 2-6, 2006

THE RIJKE TUBE: GREEN'S FUNCTION AND STABILITY ANALYSIS

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Abstract

A theoretical model is presented to predict the stability of a Rijke tube from a Green's function analysis. The Rijke tube is modelled as an open-ended tube with a blockage (compact compared with the wavelength) and a jump in cross-sectional area. Its Green's function is calculated for certain source and observer positions. In the region near the blockage the acoustic flow can be regarded as incompressible; on either side of the blockage, acoustic wave propagation is assumed. The velocity potentials of the incompressible and acoustic regions are matched by continuity of pressure and volume flow. A linear heat release characteristic (relating perturbations in heat release to those of velocity) is considered. Oscillations in the Rijke tube are modelled by an integral equation involving the Green's function. The growth rates are predicted from this integral equation and given in terms of properties of the heat release and of the Green's function.

INTRODUCTION

We consider a Rijke tube with axisymmetric geometry; a cross-section between the tube axis and the tube wall is shown in Figure 1. The ends are open with pressure nodes just outside the tube at $x=\ell_1$ and $x=\ell_2$ (Rayleigh end correction). There is a blockage, a change of cross-sectional area from $\mathcal{A}_1$ to $\mathcal{A}_2$ and a jump in mean temperature from $\bar{T}_1$ to $\bar{T}_2$. The speed of sound jumps from c_1 to c_2 due to the temperature jump.

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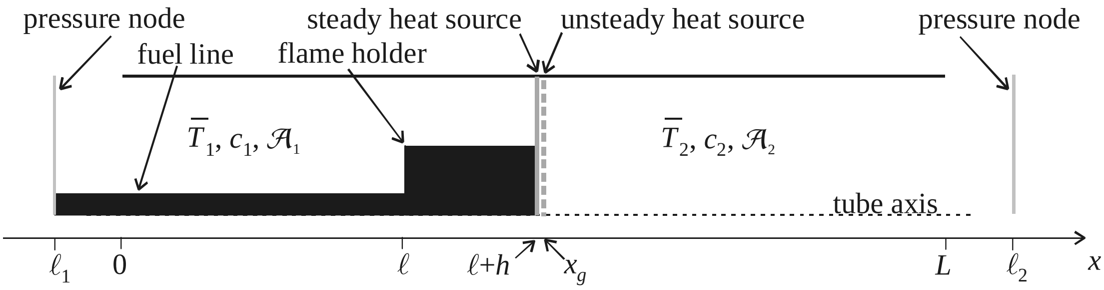

Figure 1 - Rijke tube with a point source at $x = x_g$

The jump in mean temperature is caused by a steady heat source (marked by a solid grey line) which is situated near the downstream edge of the flame holder, at $x = \ell + h$; the unsteady heat source (marked by a broken grey line) is assumed to be just downstream of this point, at $x = x_g$.

THE GREEN'S FUNCTION

An important component of our theoretical model is the exact acoustic Green's function $G(\mathbf{x}, \mathbf{x}', t, t')$. This is the velocity potential in the tube at position $\mathbf{x}$ and time t, created by an impulsive point source at position $\mathbf{x}'$ and time t'. The exact Green's function is the solution of

$$\frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} - \nabla^2 G = \delta(\mathbf{x} - \mathbf{x}')\delta(t - t'), \tag{1}$$

inside the tube. It is zero at $x = \ell_1$ and $x = \ell_2$; this neglects losses from the ends. It has a normal derivative equal to zero on all internal surfaces and on the tube axis; it also satisfies the conditions of reciprocity and causality.

For the calculation of the exact Green's function we divide the tube up into three regions: a hydrodynamic region which surrounds the flame holder and two acoustic regions (one upstream and one downstream) on either side of the hydrodynamic region. In the hydrodynamic region the field is three-dimensional. This region is small compared with the wavelength of low modes, so the acoustic motion can be treated as incompressible in this region. In the acoustic regions, the field is one-dimensional and acoustic waves are assumed to propagate in these regions. The velocity potentials in the three regions are matched at the interfaces between them by assuming continuity of pressure and volume flow across the interfaces.

We have calculated the analytic approximation to the Green's function, focussing on the simple case where the source and the observer are in the downstream acoustic region. This involved solving the one-dimensional frequency-domain version of equation (1) in the acoustic regions, solving Laplace's equation $\nabla^2 G = 0$ in the hydrodynamic region, and matching the solutions at the interfaces. Subsequent integration with respect to frequency gave the exact Green's function in the time domain. This has the form

$$G(x, x', t-t') = \sum_{n=1}^{\infty} g_n(x, x') \mathrm{H}(t-t') \sin \omega_n (t-t'), \tag{2}$$

where ω_n are the eigenfrequencies of the Rijke tube, g_n are the corresponding amplitudes, and H is the Heaviside function. This Green's function behaves as expected from an impulse response of a finite system: it is zero before the impulse (at $t = t'$) and a superposition of eigenmodes thereafter.

The eigenfrequencies ω_n are the roots of $f(\omega) = 0$ (see [1]), where

$$f(\omega) = \frac{1}{c_1}\left[-\cos\omega\tau_1 \sin\omega\tau_2 + \frac{\mathcal{A}_2}{\mathcal{A}_1}\frac{c_1}{c_2} \sin\omega\tau_1 \cos\omega\tau_2 + \frac{L_B}{c_2}\omega \cos\omega\tau_1 \cos\omega\tau_2 \right], \tag{3}$$

with $\tau_1 = \dfrac{\ell - \ell_1}{c_1}$ and $\tau_2 = \dfrac{\ell - \ell_2}{c_2}$. L_B is the "blockage integral"; this depends on the geometry of the flame holder and has to be calculated numerically (see [4]). L_B can be thought of as the length of an incompressible air-plug which fills the gap between the flame holder and the tube wall and oscillates parallel to the x-axis.

The Green's function amplitudes in (2) are given by (see [2])

$$g_n(x, x') = 2\frac{\mathcal{A}_2}{\mathcal{A}_1} \frac{C(x, \omega_n) D(x', \omega_n)}{\omega_n f'(\omega_n)}, \tag{4}$$

where f' is the derivative of the function $f(\omega)$ in (3); C and D give the x-dependence and x'- dependence, respectively, of g_n

$$C(x, \omega) = \sin\frac{\omega(x - \ell_2)}{c_2}, \tag{5}$$

$$D(x', \omega) = \cos\frac{\omega(x' - \ell)}{c_2} \sin\omega\tau_1 + \frac{\mathcal{A}_1}{\mathcal{A}_2}\frac{c_2}{c_1} \sin\frac{\omega(x' - \ell)}{c_2} \cos\omega\tau_1 + \\ + \frac{\mathcal{A}_1}{\mathcal{A}_2}\frac{L_B}{c_1}\omega \cos\frac{\omega(x' - \ell)}{c_2} \cos\omega\tau_1 . \tag{6}$$

THE GOVERNING EQUATIONS FOR THE STABILITY PROBLEM

The velocity potential ϕ in the Rijke tube is governed by the nonhomogeneous wave equation (see [3] p.508),

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}=-\frac{\gamma-1}{c^2}q'(x,t)\,, \tag{7}$$

where q' is the fluctuating part of the rate of heat release per unit mass of air (from the heat source to the air), and γ is the specific heat ratio; the speed of sound c takes values c_1 in the downstream region and c_2 in the upstream region.

In our configuration, the heat release is concentrated at the axial position x_g. We assume that its heat release rate q' depends linearly on the velocity fluctuation u' at x_g, and that there is a time lag τ between these fluctuating quantities,

$$q'(x,t)=\frac{c^2}{\gamma-1}C_\beta u'(x,t-\tau)\delta(x-x_g)\,. \tag{8}$$

This heat release characteristic is known to apply to hot gauzes (see [3] p. 511) and to some simple flames. C_β is a measure of the strength of the heat source ($C_\beta>0$). With $u'=\dfrac{\partial\phi}{\partial x}$, and equations (7) and (8), we obtain the governing equation

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}=-C_\beta\frac{\partial\phi(x,t-\tau)}{\partial x}\delta(x-x_g)\,. \tag{9}$$

Instead of solving this equation directly, we will derive an integral equation involving the Green's function.

To this end, equation (9) and the one-dimensional form of (1) are written in terms of the source variables x' and t'; (1) is multiplied by $\phi(x',t')$, (9) by $G(x,x',t-t')$, and the resulting equations are subtracted. This gives

$$\phi(x',t')\delta(x-x')\delta(t-t')+C_\beta G\frac{\partial\phi(x',t'-\tau)}{\partial x'}\delta(x'-x_g)=$$
$$=\frac{1}{c^2}\left(\phi\frac{\partial^2 G}{\partial t'^2}-G\frac{\partial^2\phi}{\partial t'^2}\right)-\left(\phi\frac{\partial^2 G}{\partial x'^2}-G\frac{\partial^2\phi}{\partial x'^2}\right). \tag{10}$$

This is integrated with respect to x' (from ℓ_1 to ℓ_2) and t' (from 0 to t). The result can be simplified (using the boundary conditions at the tube ends and causality, see [2]) to give an integro-differential equation for the velocity potential ϕ,

$$\phi(x,t) = -C_\beta \int_{t'=\tau}^{t} G(x,x_g,t-t') \frac{\partial\phi(x',t'-\tau)}{\partial x'} \Big|_{x'=x_g} \mathrm{d}t', \tag{11}$$

where initial conditions of zero initial velocity and acceleration have been assumed throughout the tube. This can be turned into an integral equation for the velocity by differentiating with respect to x. Also, the integral can be made to start at $t'=0$ by use of the Heaviside function. Evaluation at $x = x_g$ leads to an integral equation of the Volterra type for the velocity at the heat source,

$$u_g(t) = -C_\beta \int_{t'=0}^{t} \frac{\partial G(x,x',t-t')}{\partial x} \Bigg|_{\substack{x=x_g \\ x'=x_g}} \mathrm{H}(t'-\tau) u_g(t'-\tau)\mathrm{d}t', \tag{12}$$

where the abbreviation $u_g(t) = u'(x_g,t)$ has been introduced.

SOLUTION OF THE GOVERNING INTEGRAL EQUATION

We assume that u_g is a superposition of modes for $t > \tau$ with amplitudes u_m and complex eigenfrequencies $\Psi_m = \Omega_m + \mathrm{i}\Delta_m$ (Ω_m is the real eigenfrequency of mode m in the tube with unsteady heating),

$$u_g(t) = \sum_{m=1}^{\infty} \left(u_m \mathrm{e}^{-\mathrm{i}\Psi_m t} + u_m^* \mathrm{e}^{\mathrm{i}\Psi_m^* t} \right). \tag{13}$$

We focus on the imaginary part Δ_m, which indicates whether mode m is stable ($\Delta_m \le 0$) or unstable ($\Delta_m > 0$).

In order to obtain an equation for Δ_m from (12), several steps have to be performed. They involve use of (2) and (13), application of the Laplace transform with respect to time to turn the integral in (12) into a simpler algebraic expression, inverse Laplace transform back into the time domain and comparison of the coefficients of certain functions of time. These steps are too lengthy to show here, but details can be found in [2]. The following result is obtained for the complex eigenfrequencies Ψ_m,

$$1 = -C_\beta \sum_{n=1}^{\infty} \frac{\partial g_n(x_g, x_g)}{\partial x} \frac{\omega_n e^{i\Psi_m \tau}}{\omega_n^2 - \Psi_m^2}, \qquad m = 1, 2, 3, \ldots \tag{14}$$

These are coupled equations involving the complex eigenfrequencies of all modes. An approximate analytical solution for Ψ_m can be derived from the assumptions $\Omega_m \approx \omega_n$ for $n = m$, but not for $n \neq m$ (steady and unsteady case have similar real eigenfrequencies) and $|\Delta_m| << \Omega_m$ (small growth rates). Then the denominator of the term $\frac{\omega_n e^{i\Psi_m \tau}}{\omega_n^2 - \Psi_m^2}$ in (14) is small for $n = m$, but not for $n \neq m$, and thus the m th term dominates over all the others in the sum. This sum can then be approximated by the dominant term to give

$$1 + C_\beta \frac{\partial g_m(x_g, x_g)}{\partial x} \frac{\omega_m e^{i\Psi_m \tau}}{\omega_m^2 - \Psi_m^2} = 0. \tag{15}$$

For the lower modes, it is reasonable to assume that $|\Psi_m \tau| << 1$, so that the approximation $e^{i\Psi_m \tau} \approx 1 + i\Psi_m \tau$ can be made. (15) can then be turned into a quadratic equation for Ψ_m which has the following solutions:

$$\Psi_m = \frac{1}{2}\left(i\tau C_\beta \omega_m \frac{\partial g_m(x_g, x_g)}{\partial x} \pm \sqrt{-\left(\tau C_\beta \omega_m \frac{\partial g_m(x_g, x_g)}{\partial x}\right)^2 + 4\left(\omega_m^2 + C_\beta \omega_m \frac{\partial g_m(x_g, x_g)}{\partial x}\right)} \right). \tag{16}$$

If the term under the square root is negative, the real part of Ψ_m is zero. This describes the case where the velocity rises exponentially without oscillating. We ignore this case here and assume that the square-root term in (16) represents the real part of Ψ_m. Then

$$\Delta_m = \operatorname{Im} \Psi_m = \frac{1}{2} \tau C_\beta \omega_m \frac{\partial g_m(x_g, x_g)}{\partial x}. \tag{17}$$

Δ_m has four factors, three of which are positive (τ, C_β, ω_m). Thus the sign of Δ_m is determined by the sign of the fourth factor, $\frac{\partial g_m(x_g, x_g)}{\partial x}$, and this depends on the position x_g of the heat source along the tube axis.

NUMERICAL RESULTS

The derivative $\dfrac{\partial g_m(x_g, x_g)}{\partial x}$ was evaluated numerically for a tube with the following properties:

$L = 1\ \text{m}$ (tube length), $h = 0.03\ \text{m}$, $\ell_1 = -0.014\ \text{m}$, $\ell_2 = 1.014\ \text{m}$,

$\dfrac{\mathcal{A}_2}{\mathcal{A}_1} = 1.128$, $L_B = 0.0256\ \text{m}$,

$\bar{T}_1 = 288\ \text{K}$ (room temperature), $\bar{T}_2 = 488\ \text{K}$,

$c_1 = 342\ \text{m s}^{-1}$, $c_2 = 446\ \text{m s}^{-1}$.

The source position x_g was increased in small steps of $0.001\,L$ and covered the range $0 < x_g < L$. In order to keep the distance between the steady and unsteady heat source fixed at $0.01\,L$, ℓ (see Figure 1) was also increased, in such a way that $\ell = x_g - h - 0.01\,L$. As x_g increased, the interface at $\ell + h$ between the cold and the hot region in the tube moved towards the end at L; thus the cold region increased in size, while the hot region decreased. This led to a continuous decrease in the eigenfrequencies: ω_1 decreased from $1334\ \text{s}^{-1}$ to $1031\ \text{s}^{-1}$, and ω_2 from $2668\ \text{s}^{-1}$ to $2066\ \text{s}^{-1}$.

Figure 2 shows $\dfrac{\partial g_m(x_g, x_g)}{\partial x}$ as a function of x_g. The black curve is for the fundamental mode ($m = 1$); the grey curve is for the second mode ($m = 2$). The curves indicate that mode 1 is unstable in the range $0 < x_g < 0.440\ L$, and mode 2 in the ranges $0 < x_g < 0.205\ L$ and $0.424\ L < x_g < 0.707\ L$.

These predictions are in line with the well-known observation that the fundamental mode of a Rijke tube is unstable if the heat source is in the lower half of the Rijke tube. The predicted stability behaviour for the second mode has also been observed. Of course, the same predictions have been obtained in previous studies from simpler theories, for example from a classical control volume analysis, where balance equations for mass, momentum and energy across the heat source are formulated and analysed by an eigenvalue approach. Such approaches ignore any vorticity effects and offer no scope of modelling these.

Vortices are inevitably generated at the flame holder by the flow past it. These can interact with the acoustic field and may affect the stability behaviour of the Rijke tube. The advantage of our approach is that it can be generalized to include vorticity. This requires knowledge of the exact Green's function for the case where the source and observer position are in the hydrodynamic region surrounding the flame holder.

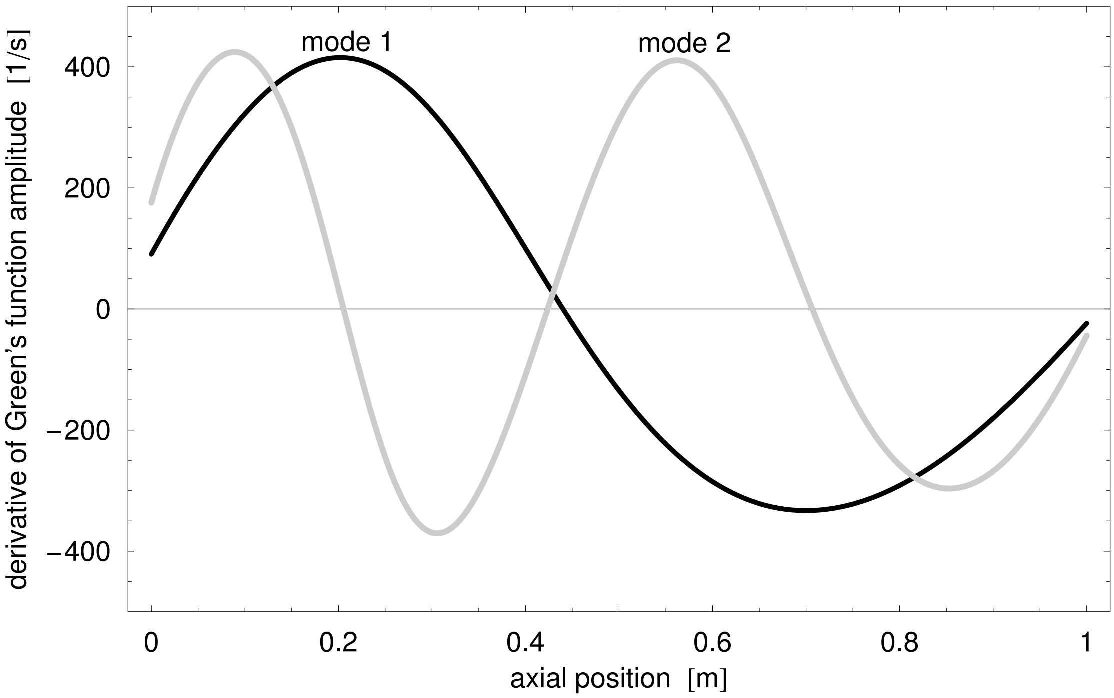

Figure 2 - Derivative with respect to x of the Green's function amplitude

CONCLUSIONS AND OUTLOOK

A Green's function approach has been used to predict the stability behaviour of a Rijke tube with a simple heat release characteristic. The predictions are in line with the well-known observation that the fundamental mode of the Rijke tube is unstable if the heat source is in the lower half of the tube. This Green's function approach offers scope to analyse Rijke tubes and Rijke burners with a variety of heat release characteristics. Also, the approach can be extended to investigate whether the vorticity generated at the flame holder has an effect on the stability behaviour.

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