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THE FREQUENCY MAPPING OF MODAL PARAMETERS IDENTIFICATION

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Abstract

When complex orthogonal polynomials (OP) are used in modal identification in frequency domain, there are two concerns, transforming coefficients between based OPs and monomials, and computing polynomials. First, the leading term coefficient of OPs usually blasts exponentially as the order increases, so is the diagonal element of the transfer matrix. This can be contributed to the fact that the orthogonal relationship among the chosen OPs is over the frequency band [0,1]. By examining the recursion of Legendre polynomials, mapping the original frequency vector into [0,2] can efficiently avoid the aforementioned exponential trend. Moreover, a numerically empirical formula of frequency mapping was proposed for sub-band fitting.

INTRODUCTION

The essence of modal parameters identification is curve fitting. Rational fraction polynomial (RFP) fitting is a frequency domain method. Before RFP fitting, the frequency vector is usually normalized, or rescaled, that is, it is mapped into a standard band to render a better numerical condition. Assuming that the frequency band before mapping is $[\omega_L, \omega_H]$. Two mappings are described in literature. One, the upper boundary after mapping is 1, that is, $[\omega_L, \omega_H]$ is mapped onto $[\omega_L/\ \omega_H, 1]$[10]. Another is mapping $[\omega_L, \omega_H]$ into $[2\omega_L/(\omega_L+\omega_H), 2\omega_H/(\omega_L+\omega_H)]$[1; 8].

Direct RFP fitting is called the Levy method[7], but its numerical condition is very bad for a high order system. To avoid ill condition in modal identification, however, orthogonal polynomial (OP) series are deliberately chosen as base functions[9-11]. This necessitates transforming between coefficients based on OPs and coefficients based on monomials. Recently, Chen et al[4] pointed out that this transform is somewhat difficult, and diagonal entries of the transfer matrix blast as the order increases. This has two potential malignancies, deteriorating the transform precision and exaggerating the error in generating the OP.

In this report, properties of the transitional matrix will be investigated. It is found that the exponential blast of diagonal entries is due that the upper boundary of the frequency vector is mapped to 1. Legendre polynomials is the asymptotic of the Forsythe polynomials[6]. By examining the recursion of this polynomial, authors find that mapping frequency vector into [0,2] can avoid the aforementioned exponential trend efficiently. Concerning the sub-band fitting, a numerically empirical formula of frequency mapping was proposed.

FITTING BASED ON MONOMIALS AND ORTHOGONAL POLYNOMIALS

For a RFP model, both the denominator and numerator are complex polynomials with real coefficients as follows:

$$\Gamma(j\omega)=\gamma_0+\gamma_1(j\omega)+\cdots+\gamma_n(j\omega)^2 \tag{1}$$

Direct fitting the model like Eq(1) is a special case of the Levy method[3; 7], with the denominator being a constant 1. It is easy to show that the normal equation is as follows,

$$[A]\{\gamma\}=\{f\} \tag{2}$$

Here $\{\gamma\}=\{\gamma_0, \gamma_1,\ldots, \gamma_n,\}^{\mathrm{T}}$ is the unknown vector to be determined. $[A]$ is $(n+1)\times(n+1)$ matrix, and the element $a_{k,l}$ is as follows

$$a_{k,l}=\begin{cases}2(-1)^{(k-l)/2}\sum_{i=1}^{N}\omega_i^{k+l} & k+l=2\kappa\\ 0 & k+l=2\kappa+1\end{cases}\qquad (k=0\sim n, l=0\sim n) \tag{3}$$

Here, N is the number of discrete frequency points. κ is a nonnegative integer. ω_i is the discrete frequency, and is distributed uniformly in an interval $[\omega_L, \omega_H]$. Denoting the frequency interval $\Delta\omega=(\omega_H-\omega_L)/(N-1)$, then $\omega_i=\omega_L+(i-1)\Delta\omega$.

In Eq(2), $\{f\}=\{f_0, f_1,\ldots,f_n\}^{\mathrm{T}}$ is the right hand side vector, and the entry f_k is

$$f_k=\begin{cases}(-1)^{k/2}\sum_{i=1}^{N}\omega_i^k h_i^R & k=2\kappa\\ (-1)^{(k+1)/2}\sum_{i=1}^{N}\omega_i^k h_i^I & k=2\kappa+1\end{cases}\qquad (k=0\sim n) \tag{4}$$

Here $h_i = h_i^{\mathrm{R}} +\mathrm{j}\, h_n^{\mathrm{I}}$ is the known or measured data at ω_i.

The matrix A in Eq(3) has a close relationship with the notorious Hilbert matrix[5; 14]. Thus, OPs are deliberately chosen as base functions [9-11] This necessitates generating OP series $p_k(s)$, k=0~n. The generation procedure was provided in several references, however, the following looks more compact,

$$\left.\begin{array}{ll}\psi_0(\omega)=1, \quad \psi_1(\omega)=\omega, & \\ \varphi_0(\omega)=\psi_0(\omega)/\nu_0, \varphi_1(\omega)=\psi_1(\omega)/\nu_1 & \\ \psi_k(\omega)=\omega\varphi_{k-1}(\omega)-\nu_{k-1}\varphi_{k-2}(\omega) & \text{for } k>1 \\ \varphi_k(\omega)=\psi_k(\omega)/\nu_k & \text{for } k\geq 0 \\ p_k(s)=p_k(j\omega)=j^k\varphi_k(\omega) & \text{for } k\geq 0\end{array}\right\} \tag{5}$$

Here, $\varphi_k(\omega)$ and $\psi_k(\omega)$ are two auxiliary polynomial function series. ν_k is a recurrent generating parameter determined by concisely

$$\nu_k=\sqrt{2\Delta\omega\sum_{i=1}^{N}|\psi_k(\omega_i)W(\omega_i)|^2} \tag{6}$$

where $W(\omega)$ is a weight function. In some references, both $\psi_k(\omega)$ and $\psi_{k+1}(\omega)$ occur in the formula of ν_k. This is not necessary. According to the orthogonal property, Eq(6) is easy to be obtained.

After estimating the coefficient vector $\{c\}=\{c_0, c_1,\ldots, c_n,\}^T$ based on the above OPs, we need transferring $\{c\}$ back to the coefficients vector $\{\gamma\}$ based on monomial. As shown in [4; 13], however, if this transform is expressed in term of recurrent generating parameters of OPs, it looks rather sophisticated. Nevertheless, the transform can be readily implemented by the coefficients matrix of OPs.

We introduce the transitional matrix

$$[\Theta]=[\theta_{l,k}]_{(n+1)\times(n+1)}=\begin{bmatrix}\theta_{0,0} & 0 & \theta_{0,2} & 0 & \theta_{0,4} & \cdots \\ 0 & \theta_{1,1} & 0 & \theta_{1,3} & 0 & \cdots \\ 0 & 0 & \theta_{2,2} & 0 & \theta_{2,4} & \cdots \\ 0 & 0 & 0 & \theta_{3,3} & 0 & \cdots \\ 0 & 0 & 0 & 0 & \theta_{4,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\end{bmatrix},$$

where

$$\theta_{l,k}=\begin{cases}\varphi_{k,l} & k\text{-}l=4\kappa \\ -\varphi_{k,l} & k\text{-}l=4\kappa+2 \\ 0 & \text{others}\end{cases} \text{ or } \theta_{l,k}=\begin{cases}p_{k,l} & k\text{-}l=2\kappa \\ 0 & \text{others}\end{cases} \tag{7}$$

Here, $\varphi_{k,l}$ is the coefficient of the term ω^l of the real polynomial $\varphi_k(\omega)$, and $p_{k,l}$ is the coefficient of the term $(\mathrm{j}\omega)^l$ in the OP $p_k(\mathrm{j}\omega)$. Both of $\varphi_{k,l}$ and $p_{k,l}$ are pure real numbers. Either $\varphi_{k,l}$ or $p_{k,l}$ can be recorded during generating OPs recurrently.

The transform between two coefficient vectors $\{\gamma\}$ and $\{c\}$ is as follows

$$\{\gamma\}=[\Theta]\{c\} \tag{8}$$

Since $\varphi_{k,l}$ is recorded during generating Ops, this transform can be realized without any problem. Noting that $[\Theta]$ is an up-triangle matrix. Moreover, this up-triangle only consists of half nonzero elements. These facts can be taken into account to save memory.

Even though coefficients matrixes of OPs [Θ] are recorded already, the efficient way to compute sampling matrixes [P] and [Q] in[4; 13] is recurrent computation by Eq(5). This is because [P] and [Q] cover data sampled from each order OPs.

With an OP base of Eq (5), the normal equation parallel to Eq (2) can be formulated. Now the normal matrix is an identical matrix, and the solution is just the right hand side vector.

ASYMPTOTIC ANALYSIS

Chen[4] found that diagonal entries of the matrix [Θ] usually increase exponentially as the order increases. This has two potential issues. One is, as stated in [4], the blasting of diagonal elements of [Θ] may deteriorate the transform precision of Eq (5). The other is the fact that the leading term become larger and larger will likely increase the possibility of the erroneous operations during generating OPs, such as adding a larger and a small number, the subtractive cancellation of 2 larger numbers. In addition, the round error occurring in computing an OP of lower order may be boosted in computing an ensuing high order OP.

An immediate idea is rescaling $\varphi_k(\omega)$ to make the leading term coefficient 1. However, the normal matrix will be not an identical matrix any longer, although it is still diagonal. Its diagonal entry sizes will increase exponentially as the order increases. This is not favourable. In fact, this does not solve the problem, but shift the predicament from the leading term to the normal matrix.

To unravel this exponential trend, we turn to the asymptotic case. Both of real functions $\psi_k(\omega)$ and $\varphi_k(\omega)$ are OPs series, if the fitting is taken over two symmetrical bands [ω_L, ω_H] and [-ω_L, $-\omega_H$], and the weight function is also reflected from [ω_L, ω_H] to [-ω_L, $-\omega_H$], as $W(\omega_i)= W(-\omega_i)$ for i=1~$\boldsymbol{N}$. For the discrete case, they are named after Forsythe polynomials[6]. If we further limit [ω_L, ω_H]=[0,1], $W(\omega_i)\equiv 1$, and $\Delta\omega\rightarrow 0$, then $\varphi_k(\omega)$s are so-called Legendre polynomial $L_k(\omega)$s. The normalized $L_k(\omega)$ has the explicit expression as following[15]

$$L_k(\omega)=\sqrt{\frac{2k+1}{2}}\times\frac{1}{2^k}\sum_{l=0}^{\lfloor k/2\rfloor}(-1)^l\binom{k}{l}\binom{2k-2l}{k}\omega^{k-2l} \tag{9}$$

Thus the leading term is

$$L_k^k=\frac{1}{2^k}\sqrt{\frac{2k+1}{2}}\binom{2k}{k}=\frac{1}{2^k}\sqrt{\frac{2k+1}{2}}\frac{(2k)!}{(k!)^2} \tag{10}$$

The ratio of leading term of two successive orders is

$$L_{k+1}^{k+1}/L_k^k=\sqrt{4-1/(k+1)^2}\approx 2 \tag{11}$$

Eq(11) shows that the leading term will increase exponentially, and increasing rate is about double. This prompts us that if the fitting frequency is dilated to [ω_L, ω_H]=[0,2], the blasting trend of the leading term will not occur any longer.

Of course, a relevant concern is that the dilation of ω_H from 1 to 2 may potentially hazard the computation of a high order OP and high order polynomials. We can argue that this concern is not necessary. Firstly, this dilation is affordable. For example, supposing that the fitting model has one hundred modals, then the leading term will be $\omega^{200} \le 2^{200} = 1.2677\times10^{30} < 10^{39} < 10^{308}$, where 10^{39} and 10^{308} are the approximate upper limits based on single precision and double precision, respectively. Therefore, this dilation generally does not lead to overflow.

Secondly, OPs have a good bound property in the fitting band, such as $|L_k(\omega)| \le \sqrt{(2k+1)/2}$. If the three-term recurrence relation is used to compute OPs, a large number, such as 2^{200}, will never appear. In fact, besides the computational efficiency forming matrixes [P] and [Q] in[4; 13] in aforementioned, directly using coefficients to compute v_k in recurrence of Eq(5) can not yield very high order OPs. So far as authors' numerical experimental experience, this recurrence will collapse at an order about 26~28 in a double precision workbench, if v_k is computed from the coefficients stored in the matrix [Θ]. This occurs even in the case of theoretically known Legendre polynomial.

Thirdly, a general polynomial can be computed by the Horner algorithm efficiently[2]. The Horner algorithm, an alternative name, Ch'in Chiu-Shao (Qin Jiu-Shao) algorithm, computes a polynomial recurrently. Thus, a large number, such as 2^{200}, will never appear in the recurrent procedure either. The overflow can only occur if the polynomial is indeed an outlier. If the function value of a polynomial on the fitting band is well bounded, the recurrent algorithm seldom confronts with the overflow problem.

MAPPING FREQUENCY VECTOR FOR PRACTICAL CASES

The above discussion applies to ω_L=0 only. In practice, piecewise fitting is used more frequently. In some extreme cases, ω_L is close to ω_H. If we still set mapping ω_H=2, the leading term will blast as the order increases either, though the blast speed is moderate. In the case of fitting real number data, mapping the fitting band into [-1,1] is recommended [12]. However, this mapping can not be implemented successfully. This is due to that the translating the origin will implicate coefficients of the fitting model with the complex number. To avert the complex coefficients, the only option is scaling.

It has already be shown above that $\varphi_{k+1,k+1}/\varphi_{k,k}$ approaches a constant for [ω_L, ω_H]= [0,2]. If we examine the trend of $\varphi_{k+1,k+1}/\varphi_{k,k}$ for $\omega_L \neq 0$, it manifest as one high followed by one low, unlike for the case of ω_L=1. However, $\varphi_{k+2,k+2}/\varphi_{k,k}$ is indeed approximate a constant. Thus, the explosion of the leading term can still be suppressed by mapping the fitting band, whereas the leading term after mapping goes up and down.

After lengthy numerical experiment, the empirical optimizing scaling factor μ for ω_L/ω_H=0~0.92 is shown in Figure 1. A good empirical model is

$$\mu = \frac{1}{\omega_H}\left[2+\left(\frac{\omega_L}{\omega_H}\right)^2\right]\exp\left[0.9\left(\frac{\omega_L}{\omega_H}\right)^6\right] \tag{12}$$

This is also shown in Figure 1. If ω_L/ω_H<0.5, a more simple scale model is

$$\mu = \frac{1}{\omega_H}\left[2+\left(\frac{\omega_L}{\omega_H}\right)^2\right] \tag{13}$$

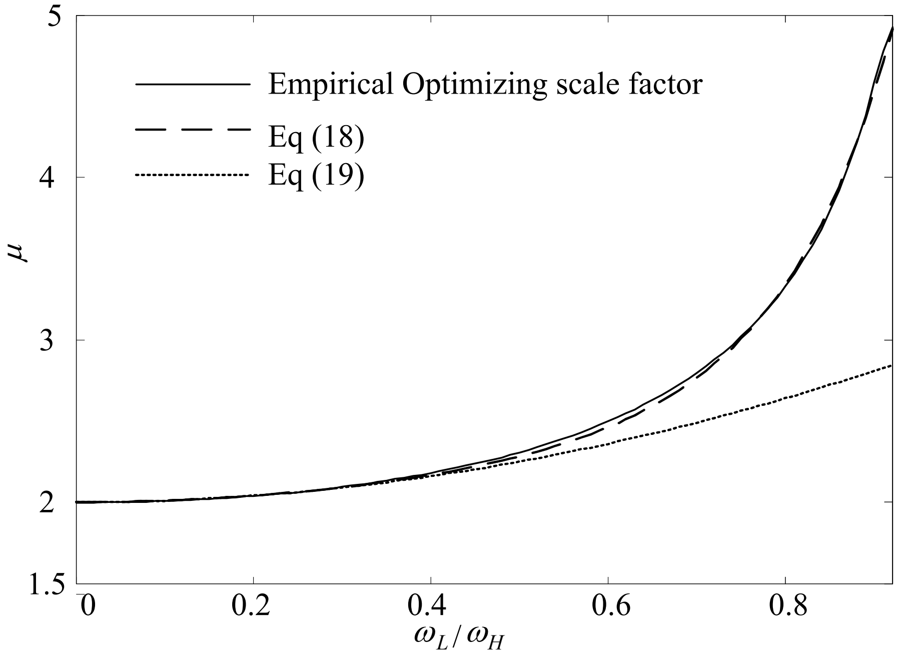

Figure 1. Optimizing scale factor

With Eq(12), we can suppress the explosion of leading term efficiently, which is instantiated in Figure 2. Here four cases are presented: case 1, [ω_L, ω_H]=[0,1]; case 2, [ω_L, ω_H]=[0.5,1]; case 3, [ω_L, ω_H]=[0.8,1]; case 4, [ω_L, ω_H]=[0.9,1]. After μ is determined from Eq(12), the fitting band is mapped into μ [ω_L, ω_H]. For all the cases, $N = 400$. Figure 2 clearly shows that the leading term before rescaling increase exponentially as the order increases. The closer is ω_L/ω_H to 1, the faster is the blasting speed. However, after rescaling, the amplitude of leading term is very steady. For ω_L/ω_H=0, leading term is almost is a constant. As ω_L/ω_H increases, the leading term coefficients manifest oscillating, one high follows by one low. Nevertheless, even for ω_L/ω_H=0.9, the oscillating amplitude is no very large.

CONCLUSIONS

Generally, frequency mapping must be conducted before curve fitting to render a better numerical condition of the normal matrix. However, if the frequency vector is mapped onto [0,1], then the leading term coefficients of the orthogonal polynomials will blast exponentially. By examining the asymptotic of the Forsythe polynomials— Legendre polynomials, we find that mapping the frequency band into [0,2] is preferable. For the practical case that the low boundary can not be mapped to zero, a new frequency mapping formula is proposed to suppress the explosive trend of the diagonal elements

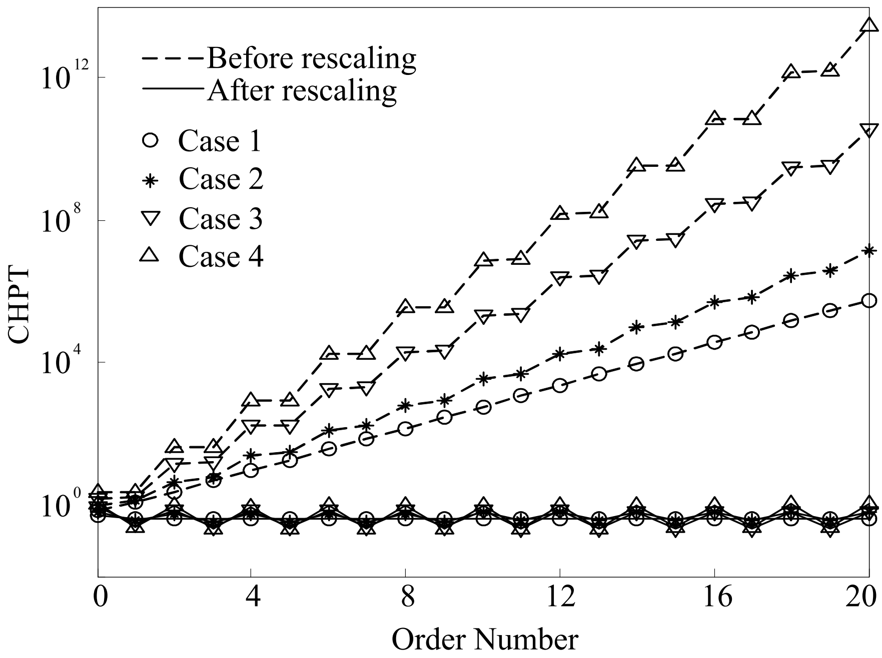

Figure 2 The sizes of the leading term coefficient leading before and after mapping

of the transitional matrix. This formula is a function of the ratio of the upper bound to the low band of the frequency band.

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