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# Eigenfrequencies Analysis of the Helmholtz problems

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**Abstract**
The new approach was used to obtain the resonant frequencies and mode shapes for the 2D/3D boundary-valued problems for the Helmholtz equation. The corresponding boundary-valued problem are solved for the different wavenumbers from some range. For the each wavenumber two integral functions were computed after obtaining numerical solution. These functions have breaks in points conterminous to eigenfrequencies. These breaks reflect influence of eigenmodes on the numerical solution. It is interesting also to note, that the number of iterations neede for convergence of conjugate gradient methods was increased unsignificantly in the neighbourhoods of eigenvalues. The comparison of numerical and analytical data in the application of the method to boundary-valued problems of the first and second kind has shown that the given method is able to find the resonant frequencies in practical problems.

## INTRODUCTION

The Helmholtz equation arises in many physical applications, in particular in acoustics. The boundary element method (BEM) is the powerful tool for the solution of many linear partial differential equations and, in particular, for the Helmholtz equation.

The traditional BEM is derived from a boundary integral equation (BIE) by dividing the domain boundary into boundary elements and applying collocation method to obtain the solution. The modern variants of BIE for the Helmholtz equation are based on the complex-valued fundamental solutions (Green's functions) in the two-dimensional (2D) and three-dimensional (3D) cases. This approach made the computational problem for the 2D/3D Helmholtz equation more complicated than, for example, for the corresponding Laplace equation. The main difficulties are connected with the computations for the high wavenumbers, when we have to use a large boundary element mesh. The memory storage is very large and CPU time is very long for solving of the large linear system with complex-valued coefficients. The modern state-of-the-art for the BEM-based solutions is shown in many reviews, in particular Brebbia [6], Banerjee and Butterfild [3], Amini and Kirkup [2].

In this work a new formulation of the boundary element method is suggested for the fast computing of the eigenfrequenciesfor the Helmholtz equation. It is a variant of the direct integral equation formulation based on the real-valued fundamental solutions [15], the application of the conjugate gradient methods for the fastsolving of the linear system and special technique for a eigenfrequencies finding.

## MATHEMATICAL FORMULATION

The Helmholtz equations for the complex-valued potential $u$ may be written in the 2D/3D space as

$$L_k^n[u] = \Delta_n u + k^2 u = 0, \tag{1}$$

where $k$ is a real-valued wavenumber; $n$ is space dimension (2 or 3). The equation is valid in the domain $D$ with boundary $S$. We consider the linear boundary-valued problems with the linear boundary conditions of the irst, second or third kind. The boundary condition on the boundary $S$ takes the following form:

$$\alpha u + \beta u_{,n} = f, \tag{2}$$

where $\alpha$, $\beta$ are real-valued coefficients; $\vec{n}$ is the unit outward normal at the point $P$ on the boundary.

Today for solving of the Helmholtz equation by the boundary element methods the fundamental solutions (Green's functions for the infinite medium) in the complex variables [1,2,3] are used as a rule. In two-dimensional case it is

$$F_{MP}^k = \frac{i}{4} H_0^{(1)}(kr_{MP}), \tag{3}$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order zero; $r_{MP}$ is a distance between the points $M$ and $P$; $k$ is a wavenumber.

In three-dimensional case the fundamental solution is

$$F_{MP}^k = \frac{1}{4\pi} \frac{e^{-ikr_{MP}}}{r_{MP}} \tag{4}$$

This approach takes a lot of memory storage and CPU time in comparison with the analogous Laplace equation. Usage of the real-valued fundamental solutions will give a reduction of memory storage in half and simplification and speeding-up program codes for computation of solution of the boundary-valued problems for the Helmholtz equation.

In the present work for two-dimensional case a new real-valued fundamental solution was used

$$F_{MP}^k = \frac{1}{4} Y_0(kr_{MP}), \tag{5}$$

where $Y_0$ is the Bessel function of the second kind of order zero (called as Neumann function of order zero also) [1]. This function is a real part of the complex-valued Hankel function of a real argument. The function has a logarithmic singularity for $r \to 0$ and the suitable conditions at $\infty$.

In a tree-dimensional case the fundamental solution was

$$F_{MP}^k = \frac{1}{4\pi}\frac{\cos kr_{MP}}{r_{MP}} \tag{6}$$

i.e., it was a real part of the corresponding complex-valued fundamental solution (4) also.

Let us $\varphi$ is a real or imaginary part of complex-valued potential $u$. Then an integral representation for $\varphi$ in domain $D$ may be written as

$$\varphi_M = \int_S (\varphi_{P,n} F_{MP} - \varphi_P F_{MP,n}) dS_P, \tag{7}$$

where point $P \in S$ and $M \in D$. Below we'll name point $M$ as viewpoint.

## BOUNDARY ELEMENT METHOD

The boundary element method is derived from the integral representation (7) by discretizing the boundary and through the application of collocation method. This approach is well known and described in details in references [1,2,5]. The discretization of (7), which leads to the classic 'boundary element method' technique is described shortly below. In the constant boundary element method, the above integral equation is solved numerically by dividing the boundary $S$ into $N$ boundary elements, in each of which $\varphi$ and $\varphi_{,n}$ are approximated by constants.

In a 2D case the boundary was approximated by straight line elements. In order that normal to the boundary outward the two nodes that define each element must be numbered in the anti-clockwise direction when it viewed from just inside the domain.

In a 3D case the boundary was represented as a set of triangular panels. In order that normal to the boundary outward the tree nodes that define each element must be numbered in the anti-clockwise direction when it viewed from just outside the domain.

We denote these values by $\varphi_i$ and $\varphi_{i,n}, i = 1, \dots N$; and apply equation (7) at one nodal point $M_i$ in the center of each boundary element to obtain

$$\frac{1}{2}\varphi_i = \sum_{j=1}^{N} (\varphi_{j,n} \int_{S_j} F_{iP} dS_P - \varphi_j \int_{S_j} F_{iP,n} dS_P) = \sum_{j=1}^{N} (A_{ij}\varphi_{j,n} + B_{ij}\varphi_j), \tag{8}$$

where $S_j$ denotes integration over the $j$th boundary element.

Coefficients $A_{ij}$ and $B_{ij}$ of the linear system of equations (8) are integrated numerically over boundary elements using the following approach.

When the viewpoint $M$ lies far from the boundary element, the integrand is continuously differentiable and hence standard quadrature rules are satisfactory. The quadrature order decreased with the increasing distance from viewpoint to the center of the boundary element.

However, when $M$ lies near the element or on the element, the integrands are became near-singular ones and standard quadrature rules are not applicable. If the viewpoint is situated near from boundary element then the special techniques applied involve 'subtracting out' the singularity and evaluating the singular part analytically and remaining regular part numerically by the same quadrature rules.

For example, for the 3D case it was

$$A_{ij} = \int_{S_j} F_{iP}^k dS_P = \int_{S_j} (F_{iP}^k - \frac{1}{4\pi r_{iP}}) dS_P + \frac{1}{4\pi} \int_{S_j} \frac{1}{r_{iP}} dS_P \tag{9}$$

$$B_{ij} = \int_{S_j} F_{iP,n}^k dS_P = \int_{S_j} [F_{iP,n}^k - \frac{1}{4\pi} (\frac{k^2}{2r_{iP}} + \frac{r_{in}}{r_{iP}^3})] dS_P + \frac{1}{4\pi} \int_{S_j} (\frac{k^2}{2r_{iP}} + \frac{r_{in}}{r_{iP}^3}] dS_P \tag{10}$$

where, in (9) and (10) the first integral is non-singular. Evaluation in this way requires the computation of the regular integral (amenable to standard quadrature) and the determination of the subtracted out part.

The regular integrals in (9, 10) that arise are approximated by a quadrature rule defined on a triangle. Paper of Laursen and Gellert [7] contains a selection of Gauss-Legendre quadrature rules for the triangle. The remaining parts in (9) and (10) are integrated analytically over boundary element.

The described procedure has allowed to develop a very exact and fast method for coefficient computation. Eliminating the $\varphi_i$ or $\varphi_{i,n}$ from each element on the boundary by applying the boundary-valued condition in each nodal point, we thus obtain from (8) a system of the $N$ linear algebraic equations with $N$ unknowns. The system of $N$ linear algebraic equations with the full matrix was solved by the different methods that described in the next section.

## SOLUTION OF A LINEAR SYSTEM

Traditionally, the direct methods have been employed to solve the resulting linear system such as the Gaussian elimination method. However, in recent years, interest has grown in iterative solvers, and in particular, the use of the conjugate gradient methods has been investigated by a number of researches ([5], [8], [9]). The field of iterative methods for solving systems of linear equations is in constant flux, with new methods and approaches continually being created, modified, tuned, and some eventually discarded. At the present work the five methods were used for the numerical solving of the linear equation system.

- CGNE for the normal equations.
- Biconjugate gradient method (BiCGJ) with Jacobi preconditioner.
- Conjugate gradient squared method.
- Biconjugate gradient stabilized method.
- Gaussian elimination method.

Algorithms of the conjugate gradient methods are described in book [4]. These methods are suitable when the matrix is asymmetric and nonsingular; however, convergence may be irregular [4], and there is a possibility that the method will break down [9]. The methods require the three or four matrix-vector multiplication at each iteration. The iterative solvers were used with Jakobi preconditioning; that is, in general, the simplest form of preconditioner.

The comparison of the different methods efficiency for the linear system was made for the 3D problems. The numerous calculations have allowed to obtain the following conclusions.

- Iteration count are relatively insensitive to mesh size. Indeed, the counts often decreased when number of the boundary elements increased.
- Iteration count are especially small for the Neumann boundary problem.
- The results are clear demonstrated the benefits from conjugate gradient methods in comparison with Gaussian elimination method.

Many of these trends are the same as for finite elements (see [8]), but the sensitive to wavenumber is considerably less pronounced.

## MODAL ANALYSIS

In this section it will be shown how the very fast boundary element method, developed in previous sections, can be used to obtain the resonant frequencies or eigenvalues of the corresponding boundary-valued problem. The problem is that of finding the values of the wavenumbers $k$ and the non-trivilar scalar functions $\phi$ such that the Helmholtz equation (1) is satisfied in an domain $D$ with boundary $S$ with the homogeneous boundary conditions of the first, second or third kind.

$$\alpha u + \beta u_{,n} = 0, \tag{11}$$

where $\alpha$ and $\beta$ are known real-valued functions.

In general, the application of the boundary element methods for the Helmholtz eigenvalue problem is not very popular, because, unlike the finite element or finite difference methods, the eigenvalue problem for BEM is nonlinear one.

The problem of solving the Helmholtz eigenvalue problem via boundary element methods have been analysed by some researchers. Firstly, the simplest idea such as the problem of finding the roots of the equation

$$\det(A_k) = 0 \tag{12}$$

was considered in works [10], [11], [12].

In figure 1 the value of determinant is presented as a function of wavenumber $k$ for the first boundary-valued problem for the interior sphere devided into 512 boundary elements. The analytical solution was $U_a = \exp ikx$. It is clear from the plot that the numerical prediction of the roots of equation (12) is a very complicated problem. It is explained to that

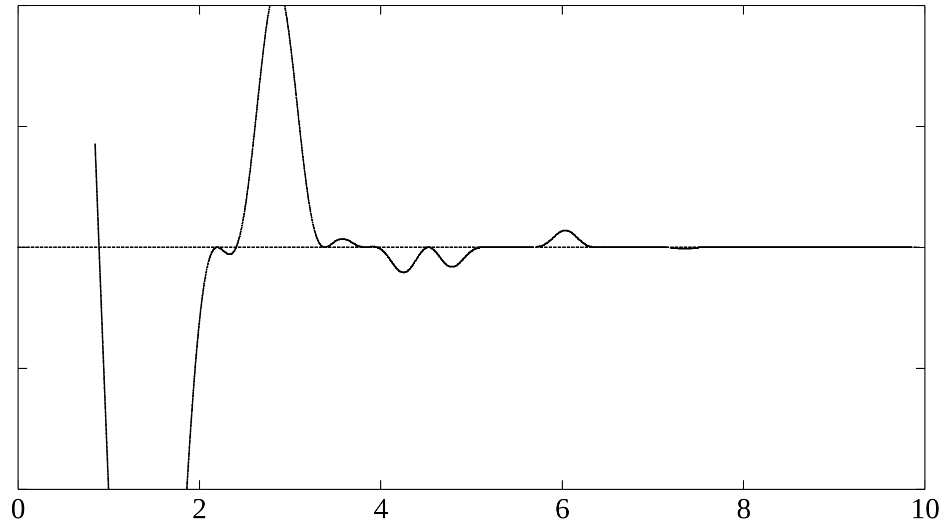


Figure 1: Determinant in 2D case as function of wavenumber

determinant computation is the more costly numerical procedure then the obtaining the solution of linear system by iterative methods.

Another approach was introduced in Kirkup and Amini [13]. The method involves approximating each component of the matrix $A_k$ by a polynomial in $k$ in some given range of wavenumber. This allows to rewrite the nonlinear eigenvalue problem in the form of a standard generalized eigenvalue problem.

In the present work for the prediction of eigenvalues the following procedure was developed. The corresponding boundary-valued problem are solved for the different wavenumbers from some range. For the each wavenumber two integral values were computed after obtaining numerical solution $u$

$$\phi = \int_S \Re u \Re u_{,n} \, dS \qquad \psi = \int_S \Im u \Im u_{,n} \, dS$$

where $\Re$ and $\Im$ are the real and imaginery parts of the complex-valued function $u$. The typical plots of $\phi$ and $\psi$ for 2D problem are shown in figure 2 as function of wavenumber $k$. It is interesting to note, that in a neighbourhoods of eigenfrequencies of function $\phi$ and $\psi$ have a discontinuity. The function $\phi$ have the nonremovable discontinuities for odd eigenfrequencies and function $\psi$ for even ones. These discontinuities reflect influence of eigenmodes on the numerical solution. It is interesting also to note, that the number of iterations neede for convergence of conjugate gradient methods was increased unsignificantly in the neighbourhoods of eigenvalues.

For check of availability of a method in a 3D case eigenfrequencies were computed in rectangular room with rigid walls and dimensions $L_x = 5$, $L_y = 4$ and $L_z = 3$. The problem has analytical eigenvalues

$$k = \sqrt{\left(\frac{i}{L_x}\right)^2 + \left(\frac{j}{L_y}\right)^2 + \left(\frac{k}{L_z}\right)^2}.$$

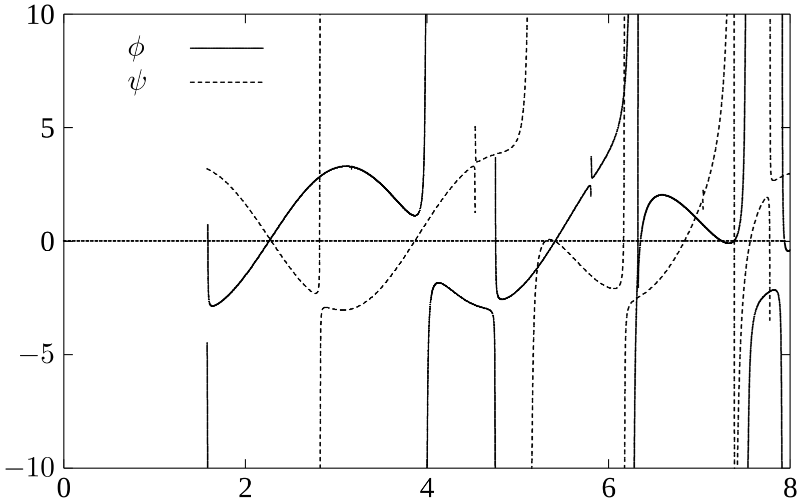


Figure 2: Values of $\phi$ and $\psi$ as functions of wavenumber $k$. Interior of sphere with boundary-valued problem of the first kind with 512 boundary elements.

The computed lowermost eigenfrequencies are submitted in the table 1 for the mesh with N=1032 and N=3008 boundary elements. From this table it is obvious, that in a case of richer

Table 1: Comparison of the numerical and analytical eigenfrequencies for rectangular room

| Analytical | Num. N=1032 | Num. N=3008 |
|---|---|---|
| 1.4520 | 1.4423 | 1.4539 |
| 1.8146 | 1.8114 | 1.8133 |
| 1.9897 | 1.9611 | 1.9832 |
| 2.2679 | 2.2668 | 2.2630 |

grid the accuracy of received results is higher.

## SUMMARY

Boundary element methods are becoming increasingly popular as methods for the numerical solution of linear elliptic partial differential equations such as the Helmholtz equation. The advantages of the proposed variant of the BEM for the 2D/3D Helmholtz equation arises from the fact that we can separately solve boundary-valued problems for the real and imaginary parts of the complex-valued potential. Use of conjugate gradient methods accelerates computations and does algorithms highly parallelizable.

The developed boundary element method can be confidently applied to the 2D/3D modal analysis problem. The comparison of numerical and analytical data in the application of the method to boundary-valued problems of the first and second kind show that the BEM is able to find the resonant frequencies in practical problems.

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