ICSV13 - Vienna
The Thirteenth International Congress on Sound and Vibration
Vienna, Austria, July 2-6, 2006

# MODAL IDENTIFICATION FROM EXPERIMENTS OF INHOMOGENEOUS STRUCTURES USING AN EXTENDED KARHUNEN-LOEVE DECOMPOSITION

Umberto Iemma*, Salvatore A. Sciuto, and Matteo Diez

Department of Mechanical and Industrial Engineering, University *Roma Tre*
via della vasca navale 79, 00146, Roma, Italy
u.iemma@uniroma3.it

**Abstract**

The paper presents recent applications of an extension of the Karhunen-Loève Decomposition (KLD) to the modal analysis of non-homogeneous structures. Specifically, the method developed by the authors, and validated in the past only on the basis of computer simulations, is here applied to the experimental analysis of a uniform steel beam sensed with two couples of different accelerometers. In this work, dimensions and weight of the sensors are such that their influence on the beam motion can not be neglected, thus imposing to consider the instrumented system as a non-uniform vibrating structure. The natural modes of the system are evaluated as the eigenfunctions (eigenvectors in the numerical approach) of the *extended* Karhunen-Loève integral operator, whose $L^2$-kernel is the time-averaged autocorrelation tensor of the displacement vector, multiplied by the density of the structure. It has been shown that, in the limit for the observation time $T$ tending to infinity and undergoing unforced free vibrations, the eigensolutions of the Karhunen-Loève operator coincide to the natural modes of vibration of the system. In practical applications, proper modal identification has to be expected if the observed vibration is representative of the motion from a statistical point of view. This is ensured if all the modes present in the motion undergo a sufficient number of periods during the acquisition time. Thus, the acquisition time has to be sufficiently long provided, of course, that damping is not too high. In addition, the statistical nature of the method makes the results of the analysis independent of the excitation used to initiate the motion, resulting in a very appealing feature. The accelerometers signals, adequately conditioned, have been acquired by means of an automatic measuring system and the numerical data have been stored for later processing. The natural frequencies are evaluated by Fourier-transforming the projection of acceleration vector onto the Karhunen-Loève eigenfunctions. The results obtained from the experimental data are compared with those computed with a FEM code, showing a good agreement. Furthermore, a preliminary estimate of the damping as a function of the frequency is obtained using a novel methodology, based on the optimal matching of the time-histories projections with an ideal damped oscillator.

## INTRODUCTION

The Karhunen-Loève Decomposition (KLD) is a statistical method for finding a base that cover the optimal distribution of energy in the dynamics of a continuum. This method initially appeared in the signal processing literature, where it was presented by Hotelling [1] in 1933 as the Principal Component Analysis (PCA). The theory behind the method was taken again and studied in depth by Kosambi [2] in 1943, by Loève [3] in 1945 and by Karhunen [4] in 1946. Since it was applied by Lumley [5] in 1967 to uncover coherent structures in turbulent flows, it has become a standard tool in turbulence studies [6], where it is also known as the Proper Orthogonal Decomposition (POD). The theory proposed by Karhunen [4] and Loève [3] is recently emerging as a powerful tool in structural dynamics and vibration. A physical interpretation of the use of the KLD in vibrations studies has been shown by Feeny *et al.* [7] and Wolter *et al.* [9]. In structural dynamics, the method consists in constructing the time-averaged spatial autocorrelation tensor of the elastic displacement field of the structure. Its spectral analysis produces a basis, as a set of orthonormal eigenfunctions (eigenvectors, in the numerical approach) and the corresponding set of eigenvalues, which represent the energy content of each mode. It has been shown (*e.g.*, Ref. [7]) that for undamped and unforced structures with constant density, the eigenfunctions given by the standard KLD coincide with the natural modes of vibration. Recently, the formulation has been extended by Iemma *et al.* [10] to the modal identification of structures with non-uniform density. In the present work, the formulation presented in Ref. [10] is used to evaluate the natural modes of an instrumented cantilever beam.

The general theory underlying the Karhunen-Loève decomposition is briefly recalled, with emphasis on its application to quasi-periodic dynamical systems with non-uniform density. The assumptions made in structural dynamics are shown and the extented KLD is outlined. A method for estimating the natural frequencies and the damping is also shown and the experimental results are presented.

## EXTENDED KARHUNEN-LOÈVE DECOMPOSITION

In structural dynamics, the method introduced by Karhunen and Loève is used to provide a basis for the *optimal* representation of the displacement vector $\mathbf{u}(\mathbf{x}, t)$ of a vibrating inhomogeneous structure. The method provides a basis which is optimal, in the energy content sense, for the representation of the displacement vector $\mathbf{u}(\mathbf{x}, t)$ in the linear combination $\mathbf{u}(\mathbf{x}, t) = \sum_{k=1}^{n} \beta_k(t)\, \boldsymbol{\varphi}_k(\mathbf{x})$, truncated to the order $n$, with $\mathbf{x} \in \mathcal{D}$

and $t \in [0, T]$.[1] The optimality condition associated to the KLD ensures that, for a given $n$, the first $n$ KLD basis functions capture, on average, more energy than any other orthonormal basis in the linear representation of the field $\mathbf{u}$ (see, *e.g.*, Holmes *et al.* [6]). It has been shown that this property is satisfied (under certain conditions) by the natural modes, provided that the formulation is embedded in the proper Hilbert space (see Iemma *et al.* [10]). In the following, the theory underlying the extension of the KLD to the modal identification of inhomogeneous structures is briefly recalled.

We assume that the dynamics of the undamped-unforced system is governed by the equation $\rho(\mathbf{x})\ \ddot{\mathbf{u}}(\mathbf{x}, t) + \mathcal{L}\,\mathbf{u}(\mathbf{x}, t) = 0$, where $\rho = \rho(\mathbf{x})$ is the structure density. Thus, the displacement vector is given by $\mathbf{u}(\mathbf{x}, t) = \sum_{k=1}^{\infty} \alpha_k(t)\ \boldsymbol{\phi}_k(\mathbf{x})$, where $\boldsymbol{\phi}_k(\mathbf{x})$ are the natural modes (linear normal modes), solution of $\mathcal{L}\,\boldsymbol{\phi}_k(\mathbf{x}) = \rho(\mathbf{x})\,\mu_k\,\boldsymbol{\phi}_k(\mathbf{x})$, with $\int_{\mathcal{D}} \rho(\mathbf{x})\,\boldsymbol{\phi}_i(\mathbf{x}) \cdot \boldsymbol{\phi}_j(\mathbf{x})\,\mathrm{d}\mathbf{x} = \delta_{ij}$. The time dependency of the solution is given by $\alpha_k(t) = a_k\ \cos(\omega_k t + \chi_k)$, where $\omega_k = \sqrt{\mu_k}$, and $a_k,\, \chi_k \in \Re$ are determined by the initial conditions.

Assuming that the displacement vector (at a given time) belongs to the Hilbert space $L^2_\rho(\mathcal{D})$, defined by the inner product $(\mathbf{f}, \mathbf{g})_\rho := \int_{\mathcal{D}} \rho(\mathbf{x})\,\mathbf{f}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x})\,\mathrm{d}\mathbf{x}$, the optimal decomposition of the vector $\mathbf{u}$ is given by the solutions of the integral problem (a complete proof of the following equation is given in Ref. [10] and, thus, not repeated here)

$$\mathcal{L}^E_{\mathbf{R}}\,\varphi(\mathbf{x}) := \int_{\mathcal{D}} \rho(\mathbf{y})\,\mathbf{R}(\mathbf{x}, \mathbf{y})\ \boldsymbol{\varphi}(\mathbf{y})\,d\mathbf{y} = \lambda\,\boldsymbol{\varphi}(\mathbf{x}). \tag{1}$$

where $\mathbf{R}(\mathbf{x}, \mathbf{y}) := \langle \mathbf{u}(\mathbf{x}, t) \otimes \mathbf{u}(\mathbf{y}, t) \rangle$ is the time-averaged autocorrelation tensor of the displacement vector $\mathbf{u}(\mathbf{x}, t)$, being $\langle \ldots \rangle := \int_0^T \ldots \mathrm{d}t$ the time-averaging operator and $\otimes$ the standard tensor product. $\mathcal{L}^E_{\mathbf{R}}$ is the extended Karhunen-Loève integral operator and the KLD optimal basis is given by its eigensolutions. It may be shown that $\mathcal{L}^E_{\mathbf{R}}$ is selfadjoint in $L^2_\rho(\mathcal{D})$, *i.e.*, $(\,\mathbf{f}, \mathcal{L}^E_{\mathbf{R}}\,\mathbf{g}\,)_\rho = (\mathcal{L}^E_{\mathbf{R}}\,\mathbf{f}, \mathbf{g}\,)_\rho$, and compact (since the kernel of Equation 1 is bounded). Hence, its eigenvalues are real and its eigenfunctions form a complete set of orthogonal functions in the above-defined Hilbert space (see *e.g.*, Ref. [13]). Under the hypothesis of undergoing unforced free vibrations and assuming an observation time $T$ tending to infinity, the Karhunen-Loève eigenfunctions coincide with the natural modes of the structure, *i.e.*, $\boldsymbol{\varphi}_k(\mathbf{x}) = \boldsymbol{\phi}_k(\mathbf{x})$ and, in addition, $\lambda_k = \frac{1}{2}\ a_k^2$ (again, see Iemma *et al.* [10]). In practical applications, proper modal identification has to be expected if the observed vibration is representative of the motion from a statistical point of view. This is ensured if all the modes present in the motion undergo a sufficient number of periods during the acquisition time. Thus, the acquisition time has to be sufficiently long provided, of course, that damping is not too high.

[1] Note that, in general, $\mathbf{x} \in \mathbb{E}^n, n = 1, 2, 3$ and $\mathbf{u}(\mathbf{x}, t) \in \mathbb{V}^m, m = 1, 2, 3$, being $\mathbb{E}^n$ an $n$-dimensional ($n = 1, 2, 3$) Euclidean point space and $\mathbb{V}^m$ an $m$-dimensional ($m = 1, 2, 3$) vector space, with $n$ not necessarily equal to $m$; consider, for instance, the case of a bending beam ($n = 1, m = 2$), or of a bending plate ($n = 2, m = 1$).

## NATURAL FREQUENCIES AND DAMPING ESTIMATE

The natural frequencies are evaluated using the following technique. First, the coefficients $\beta_k(t)$ (see previous Section) are computed as the $L^2_\rho(\mathcal{D})$-projection of the vector $\mathbf{u}(\mathbf{x},t)$ onto the $k$-th Karhunen-Loève mode, *i.e.*, $\beta_k(t) = (\mathbf{u}, \boldsymbol{\varphi}_k)_\rho := \int_\mathcal{D} \mathbf{u}(\mathbf{x},t) \cdot \boldsymbol{\varphi}(\mathbf{x})\mathrm{d}\mathbf{x}$. Then the above coefficients are Fourier-transformed. Under the hypothesis of proper modal identification (*i.e.*, under the hypothesis that $\boldsymbol{\phi}_k = \boldsymbol{\varphi}_k$) the frequency associated to the $k$-th Karhunen-Loève mode is evaluated and assumed as the natural frequency associated to the corresponding natural mode (see *e.g.*, Refs. [9] and [11]).

The estimation of the damping is conducted by approximating the $k$-th coefficient $\beta_k(t)$ with the ideal damped oscillator

$$\hat{\beta}_k(t) := \hat{a}_k e^{-\hat{\gamma}_k t} \sin(\hat{\omega}_k t + \hat{\chi}_k) \tag{2}$$

(being $\hat{\gamma}_k$ the damping associated to the $k$-th mode) and finding those parameters $\hat{a}_k, \hat{\gamma}_k, \hat{\omega}_k, \hat{\chi}_k$ that solve the problem

$$\underset{\hat{a}_k, \hat{\gamma}_k, \hat{\omega}_k, \hat{\chi}_k}{\text{Minimize}} \qquad \int_0^T |\hat{\beta}_k(t) - \beta_k(t)|^2 \mathrm{d}t. \tag{3}$$

## NUMERICAL RESULTS

In this section, we present the application of the present method to the modal analysis of a cantilever beam, instrumented with two couples of different accelerometers (see Fig. 1). Dimensions and weight of the sensors are such that their influence on the beam motion can not be neglected, thus imposing to consider the instrumented system as a non-uniform vibrating structure.

The accelerometers signals, adequately conditioned, have been acquired by means of an automatic measuring system and the numerical data have been stored in a computer device for later processing. The extended Karhunen-Loève Decomposition outlined above has been used to process the acceleration vector of the beam $\ddot{\mathbf{u}}(\mathbf{x},t)$ (for the numerical implementation we used the formulation presented in Ref. [10]). Note that, to the aim of modal analysis, we may use either displacements, velocities or accelerations, since, as easily verified, the KLD properties hold in the three cases. Two different experiments have been performed. In the first experiment, the beam motion has been initiated by a single impulse at $t = 0$ and at the beam tip. In the second, the motion has been re-sustained by a train of impulses at a random time and at a random position on the beam. Figures 2 and 3 depict the signal of the first accelerometer for the two experiments. Figure 4 shows (in logarithmic scale) the power spectral density (PSD) of the signal for the first experiment. The peaks occur at 16.8 Hz, 99.2 Hz, 281.0 Hz and 533.2 Hz.

Figure 5 shows the Karhunen-Loève modes for the two experiments. A comparison with the natural modes of the instrumented system (computed with a FEM code) is depicted, showing a remarkable agreement. In addition, it may be noted that the results

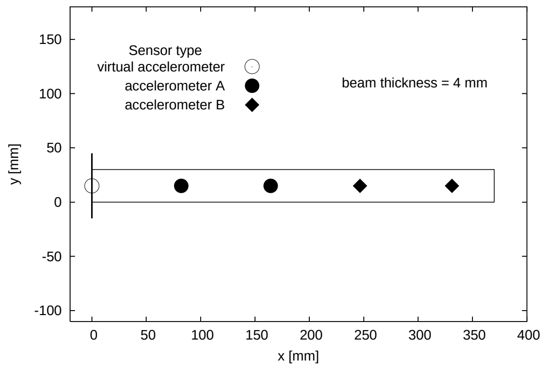


Figure 1: The instrumented beam (plant)

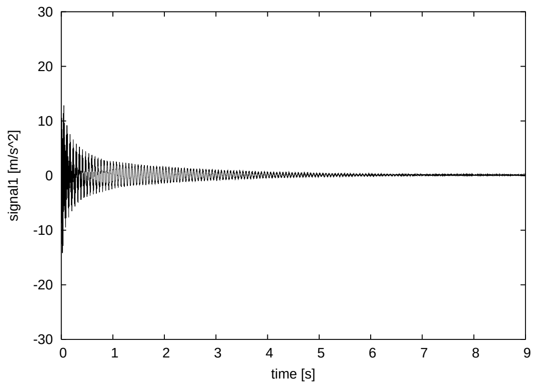


Figure 2: Signal 1 for exp. 1

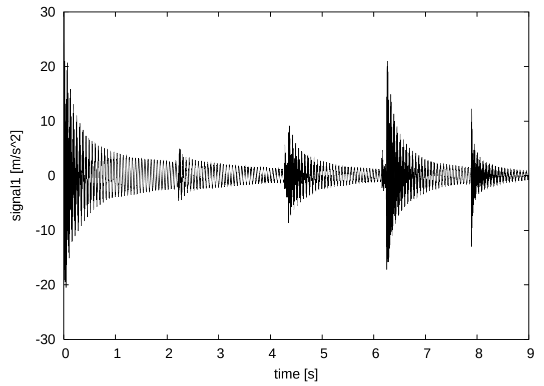


Figure 3: Signal 1 for exp. 2

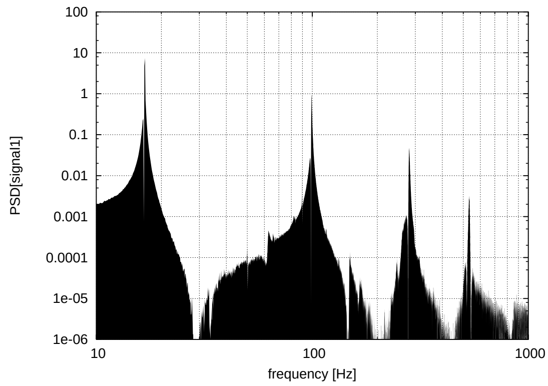


Figure 4: PSD of signal 1 (exp. 1)

of the two experiments are in excellent agreement, confirming that one may perform the KLD indipendently of the excitation used. For the first experiment, the frequency associated to each KL mode is evaluated using the method discussed above. Figures 6 and 7 show the coefficients $\beta_1(t)$ and $\beta_2(t)$ computed as the $L^2_\rho(\mathcal{D})$-projection of the acceleration vector $\ddot{\mathbf{u}}(\mathbf{x}, t)$ onto $\boldsymbol{\varphi}_1(\mathbf{x})$ and $\boldsymbol{\varphi}_2(\mathbf{x})$ respectively. Figures 8 and 9 depict (in logarithmic scale) the PSD of the above coefficients. The peaks occur respectively at 16.8 Hz and 99.3 Hz. Moreover, the same procedure, applied to the third and the fourth mode, gives the frequencies 280.9 Hz and 533.7 Hz, resulting in a very good agreement with those found in the spectral analysis of the accelerometer signal (see above). The coefficients $\beta_k(t)$ have been approximated as ideal damped oscillators as in Eq. (2). The "optimal" parameters $\hat{a}_k, \hat{\gamma}_k, \hat{\omega}_k, \hat{\chi}_k$ have been evaluated for each $\hat{\beta}_k(t)$ by solving the problem of Eq. (3). Specifically, the Broyden-Fletcher-Goldfarb-Shanno algorithm (BFGS, see, *e.g.*, Ref. [14], pp. 194-201) has been used to find the minimum of Eq. (3) in the four-variables space. In Figure 10, a comparison between the approximated model $\hat{\beta}_k(t)$ and the projection $\beta_k(t)$ is depicted for $k = 2$, showing a very good agreement (the functions coincide within plotting accuracy). The linear be-

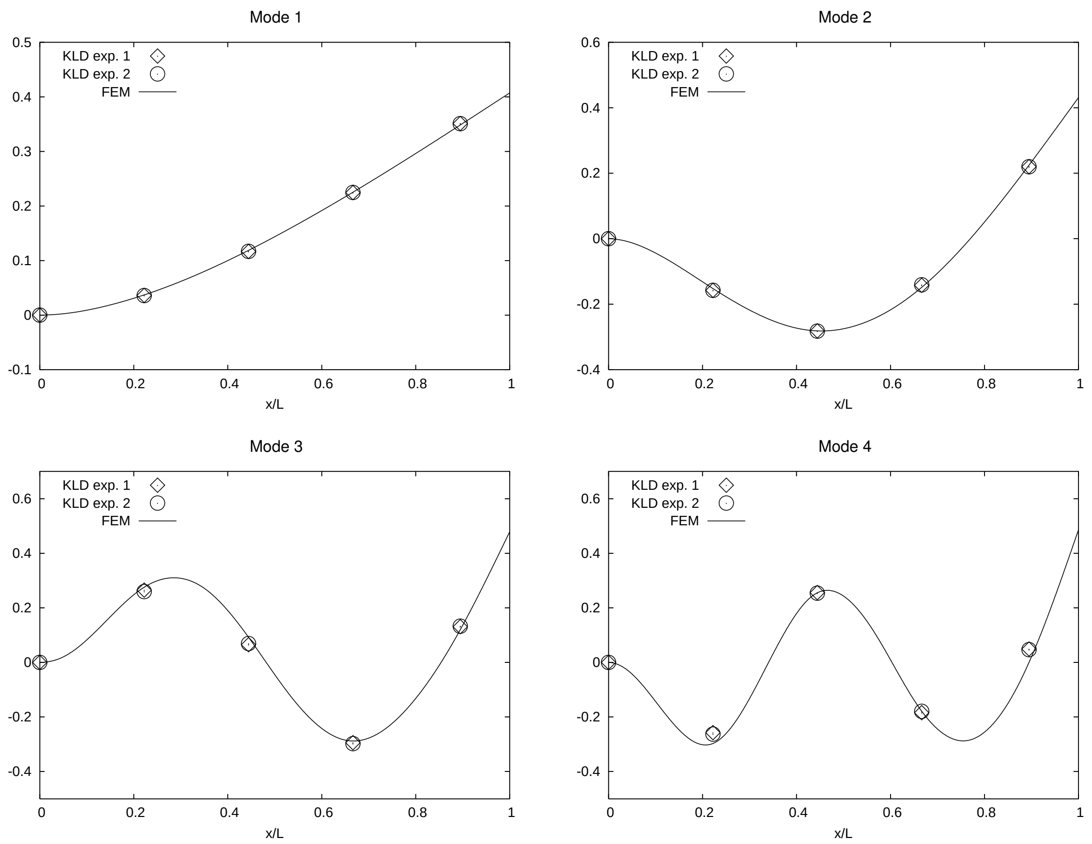


Figure 5: Comparison between Karhunen-Loève modes and natural modes (FEM)

havior of $\hat{\gamma}(f)$ (being $\hat{\gamma}$ the damping and $f$ the frequency, see Fig. 11) suggests viscous effects, such as structure-air interaction, being involved in the observed vibration.

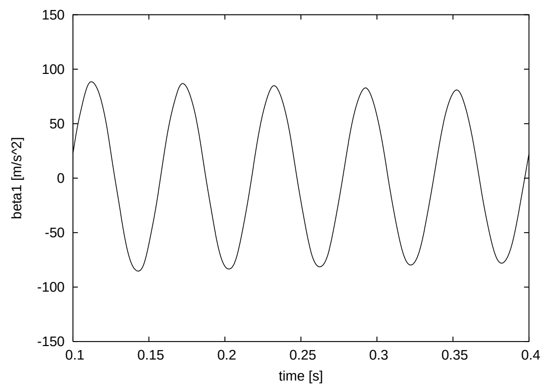


Figure 6: Coefficient $\beta_1(t)$

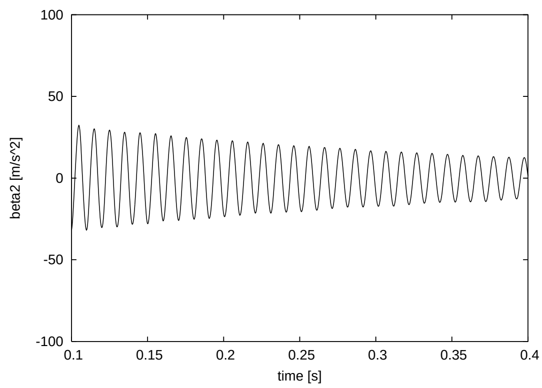


Figure 7: Coefficient $\beta_2(t)$

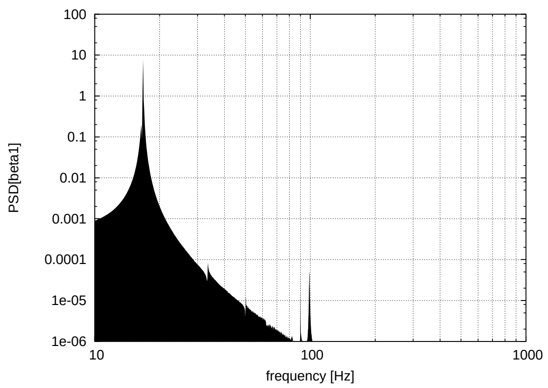


Figure 8: PSD of $\beta_1(t)$

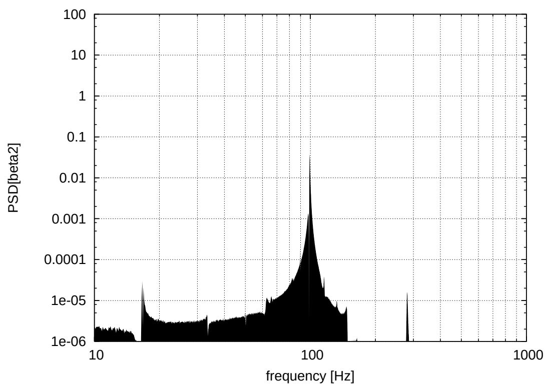


Figure 9: PSD of $\beta_2(t)$

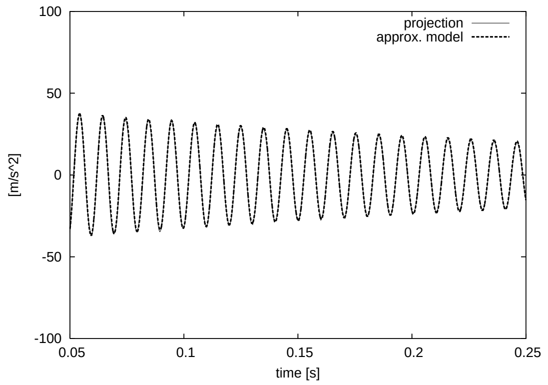


Figure 10: Approximated model for $\beta_2(t)$

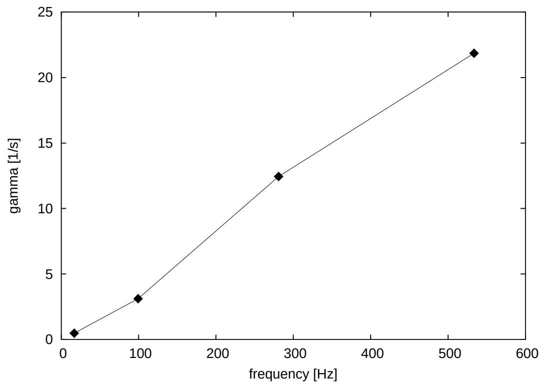


Figure 11: Damping $\hat{\gamma}$ vs. frequency $f$

## CONCLUDING REMARKS

An extension of the Karhunen-Loève Decompositon specifically aimed at the modal identification of structures with non-homogeneous density has been used to evaluate the natural modes of an instrumented cantilever beam. Two different kinds of excitation have been used during the experiments and the processing of the two sets of data gives very close results, confirming that the KLD may be performed independently of the excitation used. In addition, the results are in a good agreement with the natural modes computed with a FEM code. A method for finding the natural frequency has been used, showing a very good agreement with the results of a classical spectral analysis. A numerical technique for estimating the damping has been also presented and applied to the present case.

## Acknowledgments

The authors wish to thank Dr. Daniele Grassucci for his valuable contribution to the experimental campaign.

## References

[1] Hotelling, H., "Analysis of a complex of statistical variables into principal components," Journal of Educational Psychology, **24**, pp. 417-441 and 498-520, 1933.

[2] Kosambi, D., "Statistics in function space," J. Ind. Math. Soc., **7**, pp. 76-88, 1943.

[3] Loève, M., "Fonctions aléatoire de second ordre," Compte Rend. Acad. Sci. (Paris), **220**, 1945.

[4] Karhunen, K., "Zur Spektraltheorie stokastisher Prozesse," Ann. Acad. Sci. Fennicae, Ser. A, **1**, 1946.

[5] Lumley, J. L., "The structure of inhomogeneous turbulence," Yaglom A. M., and Tatarsky V. I., editors, *Atmospheric Turbulence and Wave Propagation*, pp. 166-178, Nauka, Moscow, 1967.

[6] Holmes, P., Lumley, J. L., Berkooz, G., *Turbulence, coherent structures, dynamical systems and symmetry*, Cambridge University Press, Cambridge, 1996.

[7] Feeny, B. F., Kappagantu, R., "On the physical interpretation of proper orthogonal modes in vibrations," Journal of Sound and Vibration, **211**, pp. 607-616, 1998.

[8] Kerschen, G., Golinval, J. C., "Physical interpretation of the proper orthogonal modes using the singular value decomposition," Journal of Sound and Vibration, **249**, pp. 849-865, 2002.

[9] Wolter, C., Trinidade, M. A., Sampaio, R., "Obtaining mode shapes through the Karhunen-Loève expansion for distributed-parameter linear systems," Shock and Vibration, **9**, pp. 177-192, 2002.

[10] Iemma U., Diez M., Morino L., "An Extended Karhunen-Love Decomposition for Modal Identification of Inhomogeneous Structures," Journal of Vibration and Acoustics, *in press*, 2006.

[11] Iemma, U., Morino, L., Diez, M., "Digital holography and Karhunen-Loève Decomposition for the modal analysis of two-dimensional vibrating structures," Journal of Sound and Vibration, **291**, pp. 107-131, 2006.

[12] Iemma, U., Diez, M., Morino, L., "Experimental modal identification of structures: The Karhunen-Loève Decomposition revisited," Eleventh International Congress on Sound and Vibration, ICSV11, St. Petersburg, 2004.

[13] Kress, R., *Linear Integral Equations*, Springer-Verlag, New York, NY, 1989.

[14] Nocedal J., Wright J., *Numerical Optimization*, Springer-Verlag, New York, NY, 2000.