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A STUDY OF THE RESONANCE FOR A CLASS OF SDOF NONLINEAR SYSTEMS

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Abstract

Based on the concept of the Nonlinear Output Frequency Response Functions (NOFRFs) proposed by authors, this paper dedicates to study the resonant phenomena of a class of nonlinear systems which can be described by a SDOF model with a polynomial type nonlinear stiffness. The concept of resonant frequencies and resonances are proposed for the first time for the nonlinear systems. This produces a novel interpretation regarding when significant energy transfer phenomena may take place with this class of nonlinear systems. The results are of significance for the design and fault diagnosis of mechanical systems and structures which can be described by the SDOF nonlinear model.

INTRODUCTION

Resonance is a well known concept. In the case of the resonances with a mechanical system, the frequency of an exciting force matches the natural frequency of the system so that the energy transmission is efficient, and the amplitude of vibration becomes significant. The resonance of linear systems has been comprehensively explored, and the knowledge about resonances of linear systems has been widely employed in many engineering practices [1]. However, in engineering there are considerable dynamical systems with nonlinear components, which cannot be simply described by a linear mode [2]-[4]. To tackle such nonlinear systems, nonlinear oscillators have been widely adopted. For example, the bilinear oscillator, piecewise linear oscillator and cubic stiffness oscillator [5] are often used to describe the changes of stiffness with working conditions. In engineering practice and laboratory research activities, resonance phenomena have been observed in nonlinear systems, which are more complicated than in linear systems. The existence of the 1/2 eigenfrequency resonance has been confirmed for a cracked object [6], and the 1/3 eigenfrequency resonance has also been observed in a system with a nonlinear stiffness [7]. Although the importance of the resonance for linear systems is well-known and the phenomena of resonances have been observed in nonlinear systems, surprisingly, there are no equivalent concepts about resonances and resonant frequencies for nonlinear systems.

Nonlinear output frequency response functions (NOFRFs) is a new concept recently proposed by the authors [8], which can be considered to be an extension of FRF of linear systems to the nonlinear case. The NOFRFs are a one dimensional function of frequency, which allows the analysis of nonlinear systems to be implemented in a manner similar to the analysis of linear system frequency responses and provides great insight into the mechanisms which dominate many nonlinear behaviours. In the present study, based on the concept of NOFRFs, the phenomena of resonance are studied for a class of nonlinear systems which can be described by a SDOF model with a polynomial type nonlinear stiffness.

NONLINEAR SYSTEMS WITH POLYNOMIAL TYPE STIFFNESS

In engineering, there are abundant dynamical systems with nonlinear components, most of which can be described as a single degree-of-freedom (SDOF) system with different nonlinear spring characteristics as shown in Equation (1) [5].

$$m\ddot{x}(t) + c\dot{x}(t) + s(x(t)) = f_0(t) \tag{1}$$

In Equation (1), m and c are the object mass and damping coefficient respectively; $x(t)$ is the displacement, and $s(x)$ is the restoring force which is a nonlinear function of $x(t)$. Some most commonly used nonlinear restoring force representations can be found in reference [5], including the bilinear stiffness oscillator often used to model a crack in a beam [9], and the piecewise linear oscillator that is able to model the connection between a control surface and servoactuator in an aircraft wing [10], and so on.

In mathematics, the Weierstrass Approximation Theorem [11] guarantees that any continuous function on a closed and bounded interval can be uniformly approximated on that interval by a polynomial to any degree of accuracy. Therefore, a SDOF nonlinear system whose restore force $s(x)$ is a continuous function of displacement x can be described by a polynomial type nonlinear system, as

$$m\ddot{x} + c\dot{x} + \sum_{i=1}^{n} k_i x^i = f_0(t) \tag{2}$$

where n is the order of the approximating polynomial, and k_i, $(i = 1, \cdots, n)$ are the characteristic parameters of the restoring force $s(x)$. In practice, many most commonly used nonlinear restoring force representations are a continuous function of displacement x, and so can essentially be regarded as a polynomial type nonlinear system. Therefore, an investigation of the polynomial type nonlinear systems will be significant for understanding complicated nonlinear phenomena in mechanical systems and structures.

The Volterra series theory of nonlinear systems is the basis of the study of a wide class of nonlinear systems including the polynomial type nonlinear system given by Equation (2). The concepts of NOFRFs are the frequency domain representations of the nonlinear systems which can be described by a Volterra series model. The NOFRFs can provide a convenient way to analyze the resonance phenomena of nonlinear systems.

NONLINEAR OUTPUT FREQUENCY RESPONSE FUNCTIONS

Consider the class of nonlinear systems which are stable at zero equilibrium and which can be described in the neighbourhood of the equilibrium by the Volterra series

$$y(t) = \sum_{n=1}^{N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \ldots, \tau_n) \prod_{i=1}^{n} u(t-\tau_i) d\tau_i \qquad (2)$$

where $h_n(\tau_1, \ldots, \tau_n)$ is the nth order Volterra kernel, and N denotes the maximum order of the system nonlinearity. The expression for the output frequency response of this class of nonlinear systems to a general input can be found in [8]. The result is

$$\begin{cases} Y(j\omega) = \sum_{n=1}^{N} Y_n(j\omega) \quad \text{for } \forall \omega \\ Y_n(j\omega) = \dfrac{1/\sqrt{n}}{(2\pi)^{n-1}} \times \int_{\omega_1+\ldots+\omega_n=\omega} H_n(j\omega_1, \ldots, j\omega_n) \prod_{i=1}^{n} U(j\omega_i) d\sigma_{n\omega} \end{cases} \qquad (3)$$

This expression reveals how nonlinear mechanisms operate on the input spectra to produce the system output frequency response. In Equation (3), $Y_n(j\omega)$ represents the nth order output frequency response of the system.

$$H_n(j\omega_1, \ldots, j\omega_n) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} h_n(\tau_1, \ldots, \tau_n) e^{-j(\omega_1\tau_1+,\ldots,+\omega_n\tau_n)} d\tau_1 \ldots d\tau_n \qquad (4)$$

is the multidimensional Fourier Transform of $h_n(\tau_1, \ldots, \tau_n)$ and is called as the Generalised Frequency Response Function (GFRF).

For linear systems, the possible output frequencies are the same as the frequencies in the input. For nonlinear systems described by Equation (3), however, the relationship between the input and output frequencies is more complicated. Given the range of the input frequency, the explicit expression for the output frequency range has been given in [8].

Based on the above results for output frequency responses of nonlinear systems, a new concept known as Nonlinear Output Frequency Response Functions (NOFRF) was recently introduced in [8]. The concept was defined as

$$G_n(j\omega) = \int_{\omega_1+\ldots+\omega_n=\omega} H_n(j\omega_1, \ldots, j\omega_n) \prod_{i=1}^{n} U(j\omega_i) d\sigma_{n\omega} \Bigg/ \int_{\omega_1+\ldots+\omega_n=\omega} \prod_{i=1}^{n} U(j\omega_i) d\sigma_{n\omega} \qquad (5)$$

under the condition that

$$U_n(j\omega) = \int_{\omega_1+\ldots+\omega_n=\omega} \prod_{i=1}^{n} U(j\omega_i) d\sigma_{n\omega} \neq 0 \qquad (6)$$

By introducing the NOFRFs, $G_n(j\omega)$, $n = 1, \cdots N$, Equation (4) can be written as

$$Y(j\omega) = \sum_{n=1}^{N} Y_n(j\omega) = \sum_{n=1}^{N} G_n(j\omega) U_n(j\omega) \qquad (7)$$

which is similar to the description of the output frequency response of linear systems. The NOFRFs reflect a combined contribution of the system and the input to the frequency domain output behaviour. According to Equation (6), the NOFRF $G_n(j\omega)$ is a weighted sum of $H_n(j\omega_1, \ldots, j\omega_n)$ over $\omega_1 + \cdots + \omega_n = \omega$ with the weights depending on the input. Therefore $G_n(j\omega)$ can be used as alternative representation of the structural dynamical properties described by H_n. There is an effective algorithm [] available which allows the estimation of the NOFRFs to be implemented directly using system input output data.

When system (2) is subject to a harmonic input

$$u(t) = A\cos(\omega_F t + \beta) \tag{8}$$

It can be deduced that the possible frequency components of $Y_n(j\omega)$ are

$$\Omega_n = \{(-n+2k)\omega_F, k = 0,1,\cdots,n\} \tag{9}$$

Moreover, it can be known that, under a harmonic loading, the NOFRF $G_n(j\omega)$ over the *n*th order output frequency range Ω_n is equal to the GFRF $H_n(j\omega_1,...,j\omega_n)$ evaluated at $\omega_1 = \cdots = \omega_k = \omega_F$, $\omega_{k+1} = \cdots = \omega_n = -\omega_F$, that is

$$G_n(j(-n+2k)\omega_F) = H_n(\overbrace{j\omega_F,\cdots,j\omega_F}^{k},\overbrace{-j\omega_F,\cdots,-j\omega_F}^{n-k}),\quad k = 0,\cdots,n \tag{10}$$

By setting

$$\varsigma = \frac{c}{2\sqrt{k_1/m}},\quad \omega_L = \sqrt{\frac{k_1}{m}},\quad \varepsilon_i = \frac{k_i}{k_1}\quad (i = 2,\cdots,n),\quad f_0(t) = \frac{f(t)}{m}$$

the polynomial type nonlinear system (2) can be expressed in a standard form

$$\ddot{x} + 2\varsigma\omega_L\dot{x} + \omega_L^2 x + \sum_{i=2}^{n}\varepsilon_2\omega_L^2 x^n = f_0(t) \tag{11}$$

The first nonlinear output frequency response function can easily be determined from the linear part of Equation (28) as

$$G_1(j\omega_F) = H_1(j\omega_F) = \frac{1}{(j\omega_F)^2 + 2\varsigma\omega_L(j\omega_F) + \omega_L^2} \tag{12}$$

The NOFRFs up to 4th order can be calculated recursively using the algorithm by Billings and Peyton Jones [12] to produce the results below.

$$G_2(j2\omega_F) = H_2(j\omega_F, j\omega_F) = -\varepsilon_2\omega_L^2 H_1^2(j\omega_F)H_1(j2\omega_F) \tag{13}$$

$$\begin{aligned} G_3(j\omega_F) &= H_3(-j\omega_F, j\omega_F, j\omega_F) \\ &= \omega_L^2\left[\frac{2}{3}\varepsilon_2^2\left(\omega_L^2 H_1(j2\omega_F) + 2\right) - \varepsilon_3\right]H_1^2(j\omega_F)\,|\,H_1(j\omega_F)\,|^2 \end{aligned} \tag{14}$$

$$G_3(j3\omega_F) = H_3(j\omega_F, j\omega_F, j\omega_F) = \omega_L^2\left(2\omega_L^2\varepsilon_2^2 H_1(j2\omega_F) - \varepsilon_3\right)H_1^3(j\omega_F)H_1(j3\omega_F) \tag{15}$$

$$\begin{aligned} G_4(j2\omega_F) &= H_4(-j\omega_F, j\omega_F, j\omega_F, j\omega_F) \\ &= -\omega_L^2 H_1(j2\omega_F)\left[\varepsilon_2 H_{42}(j2\omega_F) + \varepsilon_3 H_{43}(j2\omega_F) + \varepsilon_4 H_{44}(j2\omega_F)\right] \end{aligned} \tag{16}$$

$$\begin{aligned} G_4(j4\omega_F) &= H_4(j\omega_F, j\omega_F, j\omega_F, j\omega_F) \\ &= -\omega_L^2 H_1(j4\omega_F)\left[\varepsilon_2 H_{42}(j4\omega_F) + \varepsilon_3 H_{43}(j4\omega_F) + \varepsilon_4 H_{44}(j4\omega_F)\right] \end{aligned} \tag{17}$$

where

$$H_{42}(j2\omega_F) = H_{42}(-j\omega_F, j\omega_F, j\omega_F, j\omega_F) = \omega_L^2\,|\,H_1(j\omega_F)\,|^2\; H_1^2(j\omega_F)$$

$$\times\left\{\begin{array}{l}\left[\varepsilon_2^2\omega_L^2 H_1(j2\omega_F)-\frac{1}{2}\varepsilon_3\right]H_1(j3\omega_F)\\ +\left[\varepsilon_2^2\left(\omega_L^2 H_1(j2\omega_F)+2\right)-\frac{3}{2}\varepsilon_3\right]H_1(j\omega_F)+\varepsilon_2^2 H_1(j2\omega_F)\end{array}\right\} \tag{18}$$

$$H_{43}(j2\omega_F)=H_{43}(-j\omega_F, j\omega_F, j\omega_F, j\omega_F) \\ =-\frac{3}{2}\varepsilon_2\left[\omega_L^2 H_1(j2\omega_F)+1\right]H_1^2(j\omega_F)\,|\,H_1(j\omega_F)\,|^2 \tag{19}$$

$$H_{44}(j2\omega_F)=H_{44}(-j\omega_F, j\omega_F, j\omega_F, j\omega_F)=H_1^2(j\omega_F)\,|\,H_1(j\omega_F)\,|^2 \tag{20}$$

$$H_{42}(j4\omega_F)=H_{42}(j\omega_F, j\omega_F, j\omega_F, j\omega_F) \\ =\omega_L^2\left\{\left[4\varepsilon_2^2\omega_L^2 H_1(j2\omega_F)-2\varepsilon_3\right]H_1(j3\omega_F)+\varepsilon_2^2\omega_L^2 H_1^2(j2\omega_F)\right\}H_1^4(j\omega_F) \tag{21}$$

$$H_{43}(j4\omega_F)=H_{43}(j\omega_F, j\omega_F, j\omega_F, j\omega_F)=-3\varepsilon_2\omega_L^2 H_1^4(j\omega_F)H_1(j2\omega_F) \tag{22}$$

$$H_{44}(j4\omega_F)=H_{44}(j\omega_F, j\omega_F, j\omega_F, j\omega_F)=H_1^4(j\omega_F) \tag{23}$$

RESONANCES AND RESONANT FREQUENCIES OF NOFRFS

The resonances and resonant frequencies of the NOFRF of a nonlinear system subjected to a harmonic loading are defined as follows.

Definition: For the polynomial type nonlinear system (2) subjected to harmonic input (16), the resonant frequencies of the system *n*th order NOFRF $G_n(j\omega)$ are those ω_F's which make any of $\left|G_n(j(-n+2k)\omega_F)\right|$, $k=0,1,\cdots,n$ reach a maximum, and the maxima reached are referred to as the resonances.

For $n=1$, $\left|G_n(j(-n+2k)\omega_F)\right|$, $k=0,1,\cdots,n$, are $\left|G_1(-j\omega_F)\right|$ and $\left|G_1(j\omega_F)\right|$. In order to determine the resonant frequencies, only ω_F's which make $\left|G_1(j\omega_F)\right|$ reach a maximum need to be considered. Because $\left|G_1(j\omega_F)\right|=\left|H_1(j\omega_F)\right|$, it can be known that the resonant frequency of $G_1(j\omega_F)$ is $\omega_F=\omega_L$, and the corresponding resonance is $\left|G_1(j\omega_L)\right|=\left|H_1(j\omega_L)\right|$.

For $n=2$, $\left|G_n(j(-n+2k)\omega_F)\right|$, $k=0,1,\cdots,n$, are $\left|G_2(-j2\omega_F)\right|$ $\left|G_2(0)\right|$ and $\left|G_2(j2\omega_F)\right|$. Therefore, the resonant frequencies of $\left|G_2(j\omega)\right|$ are the ω_F's which make $\left|G_2(j2\omega_F)\right|$ reach a maximum. Equation (13) implies that the resonant frequencies of $G_2(j\omega)$ are $\omega_F=\omega_L$ and $\omega_F=\omega_L/2$, and the corresponding resonances are $\left|H_2(j\omega_L, j\omega_L)\right|$ and $\left|H_2(j\omega_L/2, j\omega_L/2)\right|$.

For $n=3$, $\left|G_n(j(-n+2k)\omega_F)\right|$, $k=0,1,\cdots,n$ are $\left|G_3(-j3\omega_F)\right|$, $\left|G_3(-j\omega_F)\right|$, $\left|G_3(j\omega_F)\right|$ and $\left|G_3(j3\omega_F)\right|$. Clearly, the resonant frequencies of $G_3(j\omega)$ are the ω_F's which make $\left|G_3(j\omega_F)\right|$ and $\left|G_3(j3\omega_F)\right|$ reach a maximum. Equations (14) and (15) indicate that the resonant frequencies of $G_3(j\omega)$ are $\omega_F=\omega_L$ and $\omega_F=\omega_L/3$, and may also include $\omega_F=\omega_L/2$, and the corresponding resonances are $\left|H_3(-j\omega_L, j\omega_L, j\omega_L)\right|$, $\left|H_3(j\omega_L, j\omega_L, j\omega_L)\right|$; $\left|H_3(j\omega_L/3, j\omega_L/3, j\omega_L/3)\right|$; and $\left|H_3(-j\omega_L/2, j\omega_L/2, j\omega_L/2)\right|$, $\left|H_3(j\omega_L/2, j\omega_L/2, j\omega_L/2)\right|$.

For n = 4, $|G_n(j(-n+2k)\omega_F)|$, $k = 0,1,\cdots,n$, are $|G_4(-j4\omega_F)|$, $|G_4(-j2\omega_F)|$, $|G_4(0)|$, $|G_4(j2\omega_F)|$ and $|G_4(j4\omega_F)|$. The resonant frequencies of $G_4(j\omega)$ are these ω_F s which make $|G_4(j2\omega_F)|$ and $|G_4(j4\omega_F)|$ reach a maximum. Equations (16)-(23) imply that the resonant frequencies of $G_4(j\omega)$ are $\omega_F = \omega_L/4$, $\omega_F = \omega_L/2$ and $\omega_F = \omega_L$, and may also have $\omega_F = \omega_L/3$, and the corresponding resonances are $|H_4(j\omega_L/4, j\omega_L/4, j\omega_L/4, j\omega_L/4)|$; $|H_4(j\omega_L/2, j\omega_L/2, j\omega_L/2, j\omega_L/2)|$, $|H_4(-j\omega_L/2, j\omega_L/2, j\omega_L/2, j\omega_L/2)|$; $|H_4(j\omega_L, j\omega_L, j\omega_L, j\omega_L)|$, $|H_4(-j\omega_L, j\omega_L, j\omega_L, j\omega_L)|$; and $|H_4(j\omega_L/3, j\omega_L/3, j\omega_L/3, j\omega_L/3)|$, $|H_4(-j\omega_L/3, j\omega_L/3, j\omega_L/3, j\omega_L/3)|$.

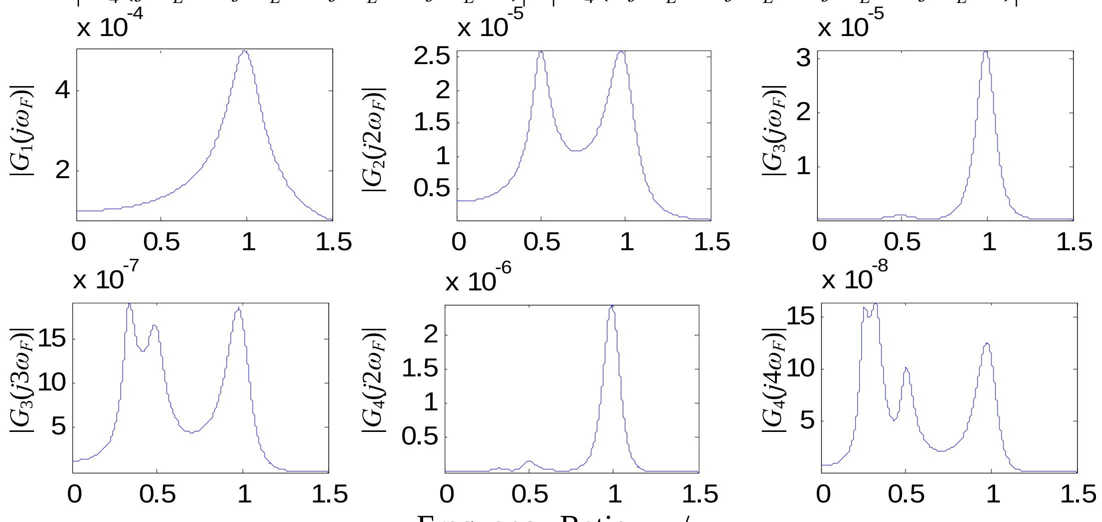


Figure 1- The NOFRFs of a nonlinear system with a 4th order polynomial type stiffness under a harmonic loading.

Figure 1 gives the NOFRFs of a nonlinear system with a 4th order polynomial type stiffness. The system parameters are $\varsigma = 0.12$, $\omega_L = 100$ rad/s, $\varepsilon_2 = 300$, $\varepsilon_3 = 5\times10^4$, $\varepsilon_4 = 9\times10^5$. Clearly, the above general analysis is confirmed by this specific example.

PHYSICAL IMPLICATION OF RESONANT FREQUENCIES OF NOFRFS

In the study of the resonance of linear mechanical systems, it is known that when the driving frequency of the force matches the natural frequency of a vibrating system, the energy transmission is efficient, and the amplitude of the vibration becomes significant. Similarly, for nonlinear system which can be described by the polynomial type nonlinear model (11), when the system subjected to a harmonic input and the driving frequency ω_F coincides with one of the resonant frequencies of a NOFRF of the system, the magnitude of this NOFRF will reach a maximum (resonance) at a high order harmonic of ω_F. Consequently, considerable input signal energy may be transferred by the system from the driving frequency to the higher order harmonic component. For example, when system (11) is subjected to a harmonic excitation with driving frequency $\omega_F = \omega_L/2$, which happens to be the resonant frequency of $G_3(j\omega)$

and $G_2(j\omega)$, a considerable input energy may be transferred through the 2nd order NOFRF and the 3rd order NOFRF from the driving frequency $\omega_L/2$ to the 2nd order harmonic component $2(\omega_L/2)=\omega_L$ and the 3rd order harmonic component $3\omega_L/2$ in the output. To demonstrate this, two harmonic inputs at the frequencies of $\omega_F=(3/2)\omega_L$ and $\omega_F=\omega_L/2$ were used respectively to excite system (11) with ς = 0.10, and the other system parameters are the same as those used in last section. As Equations (7) and (10) indicate, if $N = 4$, then the 2nd, 3rd and 4th order harmonics could appear in the system output frequency response, and the output spectrum can analytically be described as

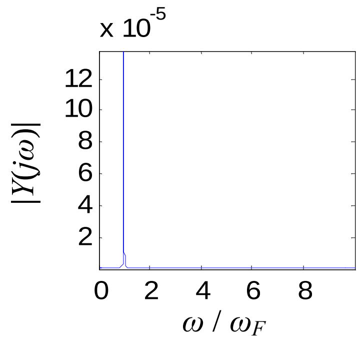
$$Y(j\omega_F)=G_1(j\omega_F)U_1(j\omega_F)+G_3(j\omega_F)U_3(j\omega_F) \tag{24}$$

$$Y(j2\omega_F)=G_2(j2\omega_F)U_2(j2\omega_F)+G_4(j2\omega_F)U_4(j2\omega_F) \tag{25}$$

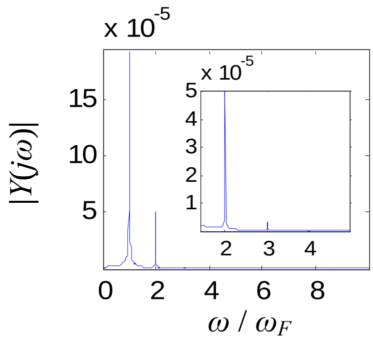
$$Y(j3\omega_F)=G_3(j3\omega_F)U_3(j3\omega_F) \tag{26}$$

$$Y(j4\omega_F)=G_4(j4\omega_F)U_4(j4\omega_F) \tag{27}$$

As frequency $\omega_L/2$ is the resonant frequency of $G_2(j\omega)$ and $G_3(j\omega)$, which could make $\left|G_2(j\omega)\right|_{\omega=2(\omega_L/2)}$ and $\left|G_3(j\omega)\right|_{\omega=3(\omega_L/2)}$ reach a maximum, according to Equation (25) and (26), it is known that the 2nd and 3rd harmonic components of the output spectrum could be considerable when $\omega_F=\omega_L/2$. In contrast, when $\omega_F=3\omega_L/2$ which is not the resonant frequencies of any of the NOFRFs involved in Equation (24)-(27), a significant high order harmonic response should not be expected in the system output. Figure (2) shows the spectra of the forced responses in the two cases of $\omega_F=\omega_L/2$ and $\omega_F=3\omega_L/2$, which were obtained by integrating equation (11) using a fourth-order *Runge–Kutta* method. It can be seen from Figures (2) that the 2nd and 3rd harmonic components in the case of $\omega_F=\omega_L/2$ are considerably more significant than in the case of $\omega_F=3/2\omega_L$.



(a) $\omega=1.5\omega_L$ (a non-resonant frequency of the involved NOFRFs)



(b) $\omega=0.5\omega_L$ (a resonant frequency of the involved NOFRFs)

Figure 2- An illustration of the physical implication of the resonant frequencies of NOFRFs

These observations lead to a novel interpretation regarding when significant energy transfer phenomena may take place with nonlinear systems subjected to a harmonic input. The interpretation is based on the concept of resonant frequencies of NOFRFs, and concludes that significant energy transfer phenomena may occur with a nonlinear system when the driving frequency of the harmonic input happens to be one of the resonances of the NOFRFs.

CONCLUSIONS

Based on the novel concept of NOFRFs, this paper dedicates to introduce the concept of resonances and resonant frequencies for SDOF nonlinear systems with a polynomial type stiffness, which can model a wide range of practical vibration components with nonlinear stiffness characteristics. Then the study gives, for the first time, the definitions of the resonances and resonant frequencies of nonlinear systems, and reveals that all higher order NOFRFs generally have more than one resonances which usually appear when the driving frequency happens to be ω_L, $1/2\omega_L$, $1/3\omega_L$, $1/4\omega_L$, and so on with ω_L being the natural frequency of the system. This is an important conclusion regarding the resonant phenomenon of the polynomial type nonlinear systems, and is of practical significance for the system design.

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