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NONLINEAR DYNAMIC ANALYSIS OF A MOTION TRANSFORMER MIMICKING A HULA HOOP

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Abstract
The motions of a hula hoop are commonly regarded as the circular oscillations where a ring undergoes around a moving human body. Based on fundamental dynamic concepts, the hula loop motions are made possible by the interactive forces between the moving ring and human body. Inspired by the generic concept of the hula hoop motion, this study proposes a novel design of a motion transformer that consists of mainly a main mass sprung in one translational direction and a free-moving mass attached at one end of a rod, the other end of which is hinged onto the main mass. It is expected that the transformer is capable of transforming linear reciprocating motions to rotary ones based on the concepts similar to the hula hoop motions. In this way, the proposed transformer could be integrated with coils, magnets, and electric circuits to form a portable power generator. To ensure the aforementioned performance of the proposed transformer, a thorough dynamic analysis on the proposed transformer dynamic system is conducted in this study to understand relation between the varied system parameters and the chance of occurrence of a hula-loop motion. The governing equations are first formulated based on dynamic principles, which is followed by the search for steady-state solutions and corresponding stability analysis via the methods of harmonic balance and Floquet theory, respectively. Based on the obtained results, the design guidelines for determining transformer parameters to ensure the occurrence of hula-loop motions are distilled.

1 INTRODUCTION

The motions of a hula hoop are commonly regarded as the circular oscillations where a ring undergoes around a moving human body. Based on fundamental dynamic concepts, the occurrence of the hula loop motions are due to the interactive forces between the moving ring and human body. Inspired by the generic concept of the hula hoop motion, this study proposes a novel design of a motion transformer that consists of mainly a main mass sprung in one translational direction and a free-moving mass attached at one end of a rod, the other end of which is hinged onto the main mass. It is expected that the transformer is capable of transforming linear reciprocating motions to rotary ones based on the concepts similar to the hula hoop motions. The proposed transformer could be integrated with coils, magnets, and electric circuits to form a portable power generator. To ensure the aforementioned performance of the proposed transformer, a thorough dynamic analysis on the proposed transformer dynamic system is conducted in this study to understand relation between the varied system parameters and the chance of occurrence of a hula-loop motion. Note that Hatwal et al. [1] presented a dynamic analysis on the system virtually owning the dynamic structure, but concentrating on the small-amplitude oscillating motions of the free mass. In this study, the free mass is expected to exhibit rotational motions as the hula-lop motions. The governing equations are first formulated based on dynamic principles, which is followed by the search for steady-state solutions and corresponding stability analysis via the methods of harmonic balance and Floquet theory, respectively. Based on the obtained results, the design guidelines for determining transformer parameters to ensure the occurrence of hula-loop motions are distilled.

2 DYNAMICAL EQUATION

Mimicking the hula hoop spun around a human body, the hula-hoop system can be schematically iluustrated as Fig. 1, in which a main mass, M, works as the main body on which a free (free) mass is rotating around an axis fixed to the main mass as the hula hoop. According to the dynamics of the hula-hoop motion, the free mass is required to spin around the pin located at the center of the main mass to which the external forces, F_{ex} and F_{ey}, applied. In a broad definition, the hula-hoop motion occurs as the free mass spins around the pin in the same direction, that is, the angular displacement of the free mass either grows or declines continuously, depending on the dynamic characteristics of the applied external force. In various applications, the hula-hoop system can be constructed in three different configurations shown in Fig. 2. Fig. 2 (a) is similar to Fig. 1, the main mass transmits the external forces from the pin to the inner surface of the free mass. Resembling the impact damper, Fig. 2(b) illustrates that the main mass drives the free mass via their contact surfaces. The phenomena of loss of contact and impact between the two masses may occur in these two configurations due to the insufficiency of the normal force. As a result, the hula-hoop motion will not happen. Fig. 2(c) is another alternative configuration of Fig. 1, which still possesses the dynamic characteristics of hula-hoop motion. The free mass and the pin on the main mass is connected by a rigid link to keep the two masses in a constant distance and to confine the free mass spinning along a circular path. Therefore, the above mentioned phenomena, loss of contact and impact, would not happen. Furthermore, since the objective of this paper is to design a motion transformer that is capable of transforming linear reciprocating motions to rotaries, we will adopt the configuration in Fig. 2(c) in the following analysis.

To the end of a precise dynamic modelling, the system shown in Fig. 2(c) is further modified by attaching the main mass with a spring and a damper, as shown in Fig. 3, to create the possibility of the reciprocating motions of the main mass In the figure, M, m, k, c, c_m, R and F_{ey} indicate the main mass, free mass, coefficient of the spring, damping capacity of the damper, the rotational damping due to the friction between the pin and hole, distance between the centers of the pin and the free mass, and the external force in y-direction, respectively. The motion degrees of freedom (DOFs) of the main mass and the free mass are represented by y and θ, respectively, which implies that the main mass reciprocating along one single translational DOF of y while the free mass rotates around the pin. In addition, the following assumptions are made to simplify the ensuing analysis without loss of generality but still complying with the purpose of the study.

1. The main mass is confined to move only in y direction.
2. The free mass is a point mass.
3. The link connecting the free mass and pin is rigid.
4. The two masses move parallel to ground, no gravity is considered. No bearing clearance exists at the rotary joint.

Lagrange's equation is next employed to derive the governing equations that describe the motions of the two masses under external excitation. The formation of kinetic energy is started with system motions in terms of the notations defined, yielding

$$\begin{aligned} T &= \frac{1}{2}\left(V_x^2+V_y^2\right)+\frac{1}{2}M\dot{y}^2 \\ &= \frac{1}{2}m\left(\left(R\dot{\theta}\sin\theta\right)^2+\left(\dot{y}+R\dot{\theta}\cos\theta\right)^2\right)+\frac{1}{2}M\dot{y}^2 \\ &= \frac{1}{2}m\left(R^2\dot{\theta}^2+\dot{y}^2+2R\dot{y}\dot{\theta}\cos\theta\right)+\frac{1}{2}M\dot{y}^2 \end{aligned} \tag{1}$$

where V_x and V_y denote the absolute velocities of the free mass in the x- and y-directions, respectively. On the other hand, the potential energy in the system caused by the motion of the spring can be expressed as $V=(1/2)ky^2$. The generalized forces of the system can be categorized into two kinds, acting on the main mass and acting on the free mass. Those on the main mass are the damping force exerted by the damper and external force, which can be simply captured by $-c\dot{y}+F_{ey}$. For the non-conservative force acting on the free mass that spins around the center pin is the rotational friction force, $-Rc_m\dot{\theta}$. With potentials, kinetic energies and generalized forces, the equations of motion of the system can be derived by applying Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)-\left(\frac{\partial L}{\partial q_k}\right)=Q_{q_k} \tag{2}$$

where $L=T-V$, T is the total kinetic energy, V is the total potentials, Q_{q_k}'s are generalized forces, and q_k's are the generalized coordinates. Having applied the Lagrange's equation as shown in Eq. (2), the equations governing the motions of the free mass and the main mass can be derived and then expressed as

$$R\ddot{\theta}+\frac{c_m}{m}\dot{\theta}+\ddot{y}\cos\theta=0 \tag{3}$$

$$(M+m)\ddot{y}+c\dot{y}+ky=F_{ey}+mR\left(\dot{\theta}^2\sin\theta-\ddot{\theta}\cos\theta\right) \tag{4}$$

To analyze the dynamic behaviours of the system, it is convenient to express Eqs. (3) and (4) in non-dimensional forms via the non-dimensional parameters defined by

$$\omega_0=\sqrt{\frac{k}{M+m}}\,,\quad \tau=\omega_0 t\,,\quad \frac{d}{dt}=\omega_0\frac{d}{d\tau}\,,\quad q=\frac{y}{R}\,,$$
$$\rho=\frac{c}{2(M+m)\omega_0}\,,\quad \mu=\frac{m}{M}\,,\quad \varepsilon=\frac{m}{M+m}=\frac{\mu}{1+\mu}\,,$$
$$F_{eq}(\tau)=\frac{F\cos(\Omega\tau)}{(M+m)\cdot R\cdot\omega_0^2} \tag{5}$$

yielding

$$q''(\tau)+2\rho q'(\tau)+q(\tau)+\varepsilon\left\{\theta''(\tau)\cos\theta(\tau)-\left(\theta'(\tau)\right)^2\sin\theta(\tau)\right\}=F_{eq}(\tau) \tag{6}$$

$$\theta''(\tau)+\varsigma_m\theta'+q''(\tau)\cos\theta(\tau)=0 \tag{7}$$

where the primes denotes the derivative with respect to the non-dimensional time τ. With the equations (6,7) in hands, approximate steady-state solutions are sought in section 3 and the associated stability analyses are conducted in section 4.

3 APPROXIMATE STEADY STATE SOLUTIONS

It is seen from Eqs. (6) and (7) that the hula-loop system considered in this study, as presented in Fig. 3, exhibits the dynamics that can be described by a set of two second-order non-linear differential equations. With system dynamic equations in hand, the next step is to seek the approximate particularly the steady state solutions that corresponds to the scenario of a reciprocating main mass and a rotational free mass. Since this solution is the desired one that provides the capability of transforming the translational, reciprocating motions to rotating ones, then enabling the designed system to fulfil the design purpose of power generation. With the assumed harmonic excitation $F_{eq}(\tau)$, shown in the last equation of Eqs. (5), it is reasonable to assume that the desired solution the steady-state solution of the main mass to be harmonic in the same frequency of excitation Ω plus a higher order components in the scales of ε, i.e.,

$$q=q_1\cos(\Omega\tau-\beta)+O(\varepsilon)\,, \tag{8a}$$

where q_1 denotes the amplitude of oscillation of the main mass. On the other hand, in order to reflect the dynamic characteristics of the hula-hoop motion, the steady-state response of the angular displacement of the rotating mass, θ, must grow or decline linearly and the variation of angular velocity θ' is small. Thus, the motion of the mass is assumed as a sum of a constant-velocity rotation, a super harmonic components and higher-order terms, i.e.,

$$\theta=\alpha\tau+u_1\cos(2\Omega\tau-\gamma)\,, \tag{8b}$$

where α=Ω and $|u_1|<1$. In addition, β and γ are the phase angles. Note that the assumption of the super harmonic components in double oscillating frequency 2Ω in Eq. (8b) originates from the fact that in the most of numerically simulated system responses, the motions of the rotating free mass exhibit strong a double-frequency super harmonic component from frequency content. Towards the end of approximating steady-state solutions, the assumed solution forms in Eqs. (8) are plugged back into system equations (6,7), and then employing the method of harmonic balancing to find appropriate values of { q_1, u_1, β, γ }. In the aforementioned process of finding approximate solutions of q_1 and ε in Eqs. (8a,8b), the Neumann's expansion [3] is utilized to expand the terms wherever it is necessary. The Neumann's expansion enables

$$\sin(u_1\cos\psi) = 2J_1(u_1)\cos\psi - 2J_3(u_1)\cos 3\psi + 2J_5(u_1)\cos 5\psi - + ... \quad (9)$$

$$\cos(u_1\cos\psi) = J_0(u_1) - 2J_2(u_1)\cos 2\psi + 2J_4(u_1)\cos 4\psi - + ... \quad (10)$$

where ψ=$2\Omega\tau-\gamma$; J_n, n = 0, 1, …., 5, is known as Bessel function of the first kind of order n .Employing Eq. (9)-(11), the terms in the bracket on the left-hand side of Eq. (6) and the last term on the left-hand side of Eq. (7) can be expanded as

$$(\theta')^2\sin\theta = \sin(\Omega\tau)(H_1) + \cos(\Omega\tau)(H_2) \quad (12)$$

$$\theta''\cos\theta = \sin(\Omega\tau)(L_1) + \cos(\Omega\tau)(L_2) \quad (13)$$

$$q''\cos\theta = \sin(2\Omega\tau)(Q_1) + \cos(2\Omega\tau)(Q_2) + \frac{1}{2}s_2\left[J_0c(p_1) + J_1s(p_4)\right] \quad (14)$$

where the expressions of the newly-introduced parameters are omitted for simplicity of presentation. As mentioned above that $|u_1|<1$ and only the first-order approximation is assumed, the Bessel function can be well approximated by

$$J_n(u_1) \cong \frac{u_1{}^n}{2^n n!}. \quad (15)$$

Substituting Eq. (8) and (12)-(15) into (6) and (7) and equating the coefficients of sine and cosine terms, the following four equations that will be solved for the four unknowns q_1, u_1, β, and γ are obtained.

$$(1-\Omega^2)q_1\cos\beta + 2\rho\Omega q_1\sin\beta - \frac{1}{2}\varepsilon\Omega^2 u_1\cos\gamma = F_{eqa}, \quad (16)$$

$$(1-\Omega^2)q_1\sin\beta - 2\rho\Omega q_1\cos\beta = \varepsilon\Omega^2, \quad (17)$$

$$4\Omega^2 u_1\cos\gamma - 2\varsigma_m\Omega u_1\sin\gamma + \frac{1}{2}\Omega^2 q_1\cos\beta = 0, \quad (18)$$

$$4\Omega^2 u_1 \sin\gamma + 2\varsigma_m \Omega u_1 \cos\gamma + \frac{1}{2}\Omega^2 q_1 \sin\beta = 0 \ . \tag{19}$$

Solving the pure algebraic equations (16-19) provides the solutions of $\{ q_1, u_1, \beta, \gamma \}$ that lead to a reciprocating main mass and a rotational free mass, following the motions prescribed by Eqs. (8a,b), respectively.

4 STABILITY ANLAYSIS

Based on basic theory of nonlinear dynamics, the steady-state solutions with the reciprocating main mass and the rotational free mass sought by Eqs. (16-19) need to be stable to be present in practice. Therefore, the stabilities of the steady-state solutions found by the solution procedure provided by the previous section is explored in order to determine if the solved steady-state solutions are present in practice. The process of stability analysis is initiated by adding small perturbations $\tilde{q}$ and $\tilde{\theta}$ to the principal lower-order parts of q and θ in Eqs. (8a,b), yielding

$$\begin{aligned} q &= q_1 \cos\left(f_q \tau - \beta\right) + \tilde{q} \\ \theta &= \alpha\tau + u_1 \cos\left(f_u \tau - \gamma\right) + \tilde{\theta} \end{aligned} \tag{20}$$

Substituting the perturbed q and θ in Eq. (20) into the equations of motion (6) and (7) yields the equations with the linear parts in terms of the perturbations $\tilde{q}$ and $\tilde{\theta}$ as

$$\tilde{q}'' + 2\rho\tilde{q}' + \tilde{q} + P_1\tilde{\theta}'' + P_2\tilde{\theta}' + P_3\tilde{\theta} = 0 \tag{21}$$

$$\tilde{\theta}'' + \varsigma_m\tilde{\theta}' + P_4\tilde{\theta} + P_5\tilde{q}'' = 0 \tag{22}$$

where the expressions of θ_b and P's are omitted for the sake of presentation simplicity. Eqs. (21) and (22) are in fact linear equations with periodic coefficients. The well-known Floquet's theory [2,3] is, therefore, next employed to investigate the stabilities of the solved steady-state solutions complying with the forms in Eqs. (8a,b). By letting $\tilde{q}' = \tilde{Q}$ and $\tilde{\theta}' = \tilde{O}$, the two second-order differential equations (21) and (22) are transformed into four first-order equations as

$$\tilde{\mathbf{x}}' = A(\tau)\tilde{\mathbf{x}} \tag{23}$$

With the expression of $A(\tau)$ in hands, the so-called transition matrix exp($A(T)$) over one period of oscillation T can be computed. The stability of each solved steady-state solution is then determined based on whether the eigenvalues of the transition matrix are located inside the unit circle in the complex plane. If all eigenvalues of the transition matrix are inside the unit circle, the solved solution is stable and possibly appears in practice. On contrary, if any eigenvalue of the transition matrix is outside the unit circle, then the corresponding solved solution is unstable and cannot be observed at steady state in practice. Figure 4 shows the stability analysis results following the aforementioned computation procedure on the transition matrix and associated eigenvalues. In this figure, the coarse curves denote the boundaries between stable and unstable solutions in the form of Eqs. (8a,b), which results from computation of the eigenvalues of the transition matrix. Between boundaries, there are three regions. Region I and III enclose unstable solutions, while Region II does stable ones. On the other hand, also shown in the figure are circles and crosses, which represent stable and unstable solutions, respectively. Note that the stabilities associated with circles and crosses are obtained from direct numerical solutions on the original governing equations (3) and (4). It can be seen that a general agreement is present between the stabilities prediction by the analytical Floquet theory and the direct numerical simulations despite few exceptions.

5 CONCLUTIONS

A thorough dynamical modeling is conducted for a motion transformer mimicking a hula loop, and further exploring feasibility of the desired steady-state solutions. Also preformed is the stability analysis on the desired solution via the Floquet theory in order to guarantee the practical existence of the solution. It is found that the desired steady-state solution that transforms the reciprocating motions of the main mass to rotary ones of the free mass does exist. The corresponding approximate mathematical expressions are also derived. Furthermore, via the analysis based on the Floquet theory the stability of the desired solution can be endured over a large set of combinations of driving frequencies and amplitudes. A general agreement is present between the stabilities prediction by the Floquet theory and the direct numerical simulations.

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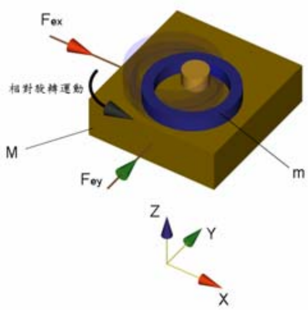

Figure 1 – Schematic of a hula-hoop system

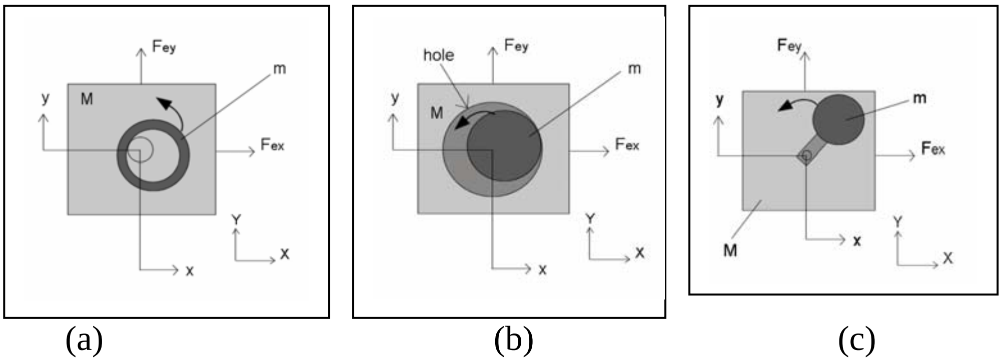

Figure 2 – Three configurations of the hula-hoop system

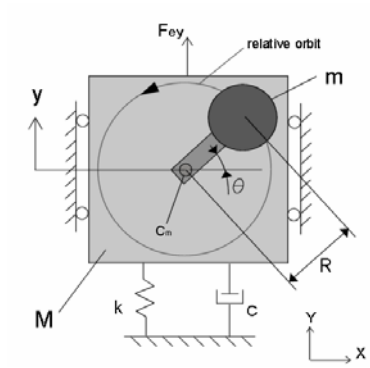

Figure 3 – Dynamic model of the entire hula-hoop system

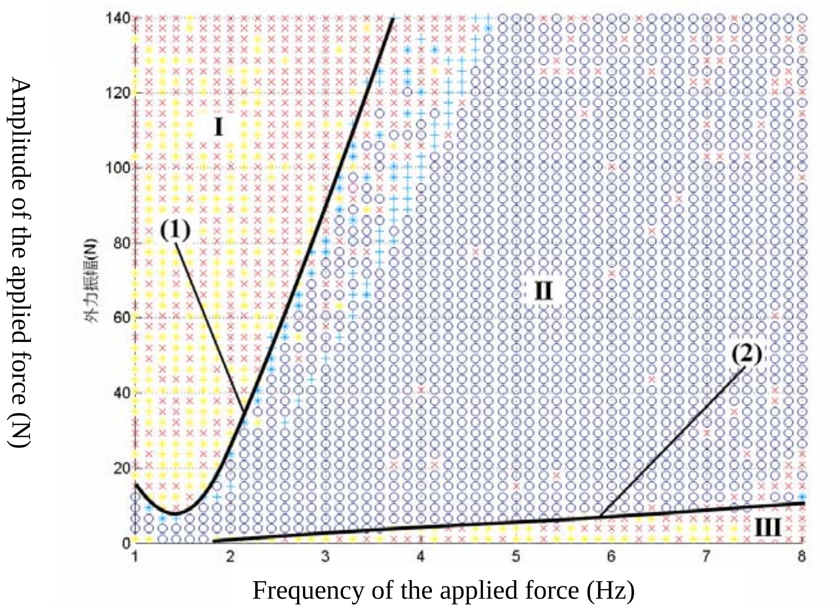

Figure 4 – Stabilities of solved steady-state solutions