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A RELIABLE ROOT FINDER FOR SYSTEMS OF COUPLED EQUATIONS: APPLICATION TO EIGENVALUES IN DUCT ACOUSTICS

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Abstract
The description of the acoustic field in lined ducts in terms of modes leads to transcendental eigenvalue equations. These can be cast into a form defining a complex function. Each zero of the function corresponds to an eigenvalue associated with a mode. A reliable method is proposed for finding zeros of complex functions within a bounded region of the complex plane. The method makes use of the Argument Principle to count the zeros of the function prior to a global Newton-Raphson iteration to find each zero. The method is guaranteed to find all zeros in the given region. The number of zeros of functions of several variables can be counted using the extension of the Argument Principle to several complex variables. The proposed method is thus suitable for solving systems of coupled eigenvalue equations in a given multi-dimensional domain. The algorithm for a coupled eigenvalue solver is described, and example applications to problems in duct acoustics are given.

INTRODUCTION

Analytic solutions for the acoustic field in a duct, expressed in terms of modes, may be found when the geometry is separable. Common examples are uniform two-dimensional ducts, rectangular-section ducts, and circular-section cylindrical or annular ducts. In such cases, the solution of the wave equation and boundary conditions may be formulated as an eigenvalue problem. Therefore, central to applications in duct acoustics is an eigenvalue solver capable of finding all eigenvalues in a region of interest. This requires the zeros of a complex function to be located. Each zero is an eigenvalue associated with a mode.

There are well established procedures for locating the required zeros of a real function of a real variable, but most are not directly applicable to complex functions. Those which are, such as the Newton-Raphson method, require a good initial estimate of the zero and do not guarantee convergence to the desired zero, or that all zeros of interest are found.

In duct acoustics problems these difficulties are usually avoided by using so called shooting methods (e.g., see ref. [1], also [2] and [3]). These consist, in one form or another, of starting an iteration with known eigenvalues, usually for rigid or pressure release boundary conditions, and tracking the eigenvalues in the complex plane as the problem parameters are successively perturbed. The procedure is iterated until the parameters reach the required values. There still remain difficulties with this process however, most notably identifying solutions associated with modes of the surface wave type, and tracking eigenvalues that come close together, i.e. near “optimum” admittance values.

One other approach follows the line of thought that a zero finding algorithm should incorporate some means of bracketing the zeros of interest within a bounded region. If, in addition, one had some means of telling exactly how many zeros fell in the region, one could certainly be satisfied that all required zeros had been found. This is the approach followed in this paper. Given some bounded region in the complex plane, count the number of zeros of the function inside the region. Once the number of zeros in the region is known, refine each zero by repeating a global Newton-Raphson iteration until all zeros are found. To avoid convergence to an already found zero, each zero is removed from the function as it is found, by collocation of a simple pole. The underlying theory and a procedure for counting zeros of complex functions is the subject of the next section. The refinement stage is described in the following section. The final section of the paper is devoted to example applications in two problems in duct acoustics.

COUNTING ZEROS OF COMPLEX FUNCTIONS

Functions of one variable

There is a classical result from complex function theory, usually referred to as the Argument Principle, which allows precisely to infer how many zeros an analytic function has within a given region. If a function $f(z)$ is analytic in some bounded region $\mathcal{D} \subset \mathbb{C}$, with no zeros on the region's boundary γ, the number of zeros of f in $\mathcal{D}$, counted with multiplicity, is given exactly by

$$N = \frac{1}{2\pi i} \int_{\gamma} \frac{f'}{f} dz. \tag{1}$$

By making the substitution $w = f(z)$, equation (1) becomes

$$N = \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{w} dw, \tag{2}$$

where $\Gamma \equiv f(\gamma)$ is the image of γ under the mapping f. Equation (2) is the definition of winding number of the contour Γ about the origin. It follows that the number of zeros of f enclosed by γ equals the number of times Γ winds around the origin. For example, the function $f(z) = z^2 - 1$ has the two zeros $z^{(1)} = -1$ and $z^{(2)} = +1$. If the contour γ encloses both zeros, then its image Γ encircles the origin twice, as shown in figure 1.

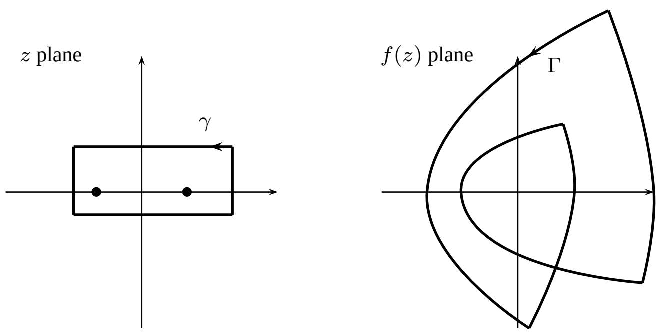

Figure 1: Contour of integration γ enclosing the two zeros of the function $f(z) = z^2 - 1$. The image contour $\Gamma \equiv f(\gamma)$ encircles the origin twice.

One fundamental property of a complex function $f(z)$, analytic in some domain $\mathcal{D}$, is that it can only have isolated zeros within $\mathcal{D}$. Furthermore, if $\mathcal{D}$ is bounded, f can have at most a finite number of zeros in $\mathcal{D}$. Consequently, the number of zeros of f in $\mathcal{D}$ must be an integer and finite quantity. This is a remarkable result, for if one has some means of approximating the integral (1) or (2) in a controlled way, it suffices to achieve an accuracy within ± 0.49 of its true magnitude to guarantee that the exact number of zeros is accounted for. If the boundary γ is a rectangle with corners labeled anti-clockwise by z_1, z_2, z_3, and z_4, it can be split into four straight paths $\gamma \equiv \gamma_1 + \cdots + \gamma_4$, each parameterized by $\gamma_i = (z_{i+1} - z_i)t + z_i$, with $0 \leq t \leq 1$, and $z_5 \equiv z_1$. The number of zeros is then given by the sum of the integrals over each image path $\Gamma_i \equiv f(\gamma_i)$. In order to guarantee that the correct number of zeros is accounted for, it is then sufficient to calculate each of these path integrals with a precision of, say 0.1.

Perhaps the simplest, effective way of approximating the integral (1) or (2) is through numerical integration, using the composite Newton-Cotes formulas, the simplest of these being the trapezoidal rule. The trapezoidal rule is readily applicable to path integrals in the complex plane and is given by

$$\int f(z)dz = S_n + R_n, \tag{3}$$

with the approximation to the integral calculated from

$$S_n = \frac{(z_1 - z_0)}{2} f(z_0) + \frac{(z_n - z_{n-1})}{2} f(z_n) + \sum_{k=1}^{n-1} \frac{(z_{k+1} - z_{k-1})}{2} f(z_k), \tag{4}$$

and the corresponding absolute value of the error $|R_n|$, for a discretization of the integration path at n equally spaced points, bounded from above by

$$|R_n| \leq \frac{|z_n - z_0|^3}{12n^2} \max |f''|, \tag{5}$$

where $\max |f''|$ is the maximum value of the second derivative of f along the path. Since the error bound decreases with the number of path sub-divisions, one can obtain arbitrarily close approximations to the true value of the integral by refining the discretization of the path. The number of zeros of a complex function can thus be determined reliably by successively applying equation (4) to finer discretizations of each of the straight paths γ_i until $|R_n|$ becomes less than 0.1. In practice, since the integrands are analytic functions, and consequently are well behaved, it suffices to refine the discretization until

$$|R_{n-1} - R_n| = |S_{n-1} - S_n| < 0.1. \tag{6}$$

Systems of functions of several variables

Consider a complex function of several complex variables $F : \mathcal{D} \subset \mathbb{C}^n \to \mathbb{C}^n$, analytic in $\mathcal{D}$, with N zeros $z^{(\nu)} \equiv \left(z_1^{(\nu)}, \ldots, z_n^{(\nu)}\right), \quad \nu = 0, \ldots, N-1$, such that

$$F\left(z^{(\nu)}\right) = \begin{cases} f_1\left(z_1^{(\nu)}, \ldots, z_n^{(\nu)}\right) = 0 \\ \vdots \\ f_n\left(z_1^{(\nu)}, \ldots, z_n^{(\nu)}\right) = 0 \end{cases}. \tag{7}$$

It can be shown that the number of zeros of the system (7) in a bounded region $\mathcal{D} \equiv \mathcal{D}_1 \times \cdots \times \mathcal{D}_n \subset \mathbb{C}^n$, counted with multiplicity, is given by (see ref. [4], pp.179)

$$N = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{dR(^*f, f_n)}{R(^*f, f_n)}, \tag{8}$$

where Γ_n is the image under f_n of the boundary γ_n of the region $\mathcal{D}_n$. The resultant $R(^*f, f_n)$ of the function f_n with respect to the system $^*f \equiv (f_1, \cdots, f_{n-1})$ is defined by

$$R\big(^*f, f_n\big) = \prod_{\nu=1}^{p} \Big[f_n\big(^*z^{(\nu)}(z_n), z_n\big) \Big]^{\mu_\nu}, \tag{9}$$

where $^*z^{(\nu)}(z_n) \equiv \big(z_1^{(\nu)}(z_n), \cdots, z_{n-1}^{(\nu)}(z_n)\big)$ are the zeros of the system $^*f$ in the bounded domain $^*\mathcal{D} \subset \mathbb{C}^{n-1}$, and μ_ν is the multiplicity of each zero $^*z^{(\nu)}$. The concept of resultant is related to the Sylvester Matrix, and has its primary use in connection with the elimination of variables from systems of polynomial equations. The above extended definition, applicable to general analytic functions, is given in [4], pp. 178.

To illustrate the concept of resultant consider for instance the function

$$F(z_1, z_2) = \begin{cases} f_1 = \sin(z_1 + z_2) \\ f_2 = z_1 \cos(z_2) \end{cases}. \tag{10}$$

Suppose that, in some bounded domain $\mathcal{D} \equiv \mathcal{D}_1 \times \mathcal{D}_2 \subset \mathbb{C}^2$, and for any given $z_2 \in \mathcal{D}_2$, f_1 has p zeros in $\mathcal{D}_1$. These are given by $z_1^{(\nu)} = \pm\nu\pi - z_2$, and are simple, so that $\mu_\nu = 1$. The resultant of f_2 with respect to f_1 is then

$$R(f_1, f_2) = \prod_{\nu=1}^{p} f_2\big(z_1^{(\nu)}, z_2\big) = \prod_{\nu=1}^{p} \big[\pm(\nu - 1)\pi - z_2\big] \cos(z_2), \tag{11}$$

a function of z_2 only. If, in $\mathcal{D}_2$, there are q values of z_2 for which $R(f_1, f_2) = 0$, there are then $p \times q$ zeros of $F(z_1, z_2)$ in $\mathcal{D}$.

In order to count the zeros of a system of complex functions of several variables it is thus sufficient to calculate the integral (8) to within ± 0.49 of its true value, as for the single variable case. With relation to a system of two functions of two variables $F(z_1, z_2) \equiv \big(f_1(z_1, z_2), f_2(z_1, z_2)\big)$, given a rectangular region $\mathcal{D}_2$, split the bounding contour γ_2 into four straight paths. For a given discretization $\big\{z_2^{(1)}, \ldots, z_2^{(n)}\big\}$ of each of the paths construct the resultant $R(f_1, f_2)$ by taking each $z_2^{(k)}$ as a parameter in the function f_1. For each $z_2^{(k)}$ count the number of zeros of $f_1\big(z_1, z_2^{(k)}\big)$ enclosed by γ_1 as for a function of the single variable z_1 and find the zeros $z_1^{(\nu)}$ using the refining procedure, which will be discussed in the beginning of the next section. The value of the resultant for each $z_2^{(k)}$ is then given by formula (9). Approximate successively the integral (8) for each path using formula (4) with increasing discretization points, until $|S_{n-1} - S_n| < 0.1$. The number of zeros $\big(z_1^{(\nu)}, z_2^{(\nu)}\big)$ of F is then the absolute value of the sum of the integrals for each of the four paths, rounded to the nearest integer. This algorithm is readily scalable to systems of any number of functions.

THE ZERO REFINING PROCEDURE

Functions of one variable

Given a randomly chosen point $z^{(0)}$ within the region of interest, a global Newton-Raphson iteration consists of calculating successive approximations to a zero of f using

$$z^{(k+1)} = z^{(k)} - \frac{f\big(z^{(k)}\big)}{f'\big(z^{(k)}\big)}. \tag{12}$$

The iteration proceeds until either $z^{(k+1)}$ becomes within a specified tolerance relative to $z^{(k)}$, or $|f\big(z^{(k+1)}\big)|$ is less than a specified absolute tolerance, or $z^{(k+1)}$ falls outside the rectangle boundary, or the number of iteration steps exceeds a specified maximum. The first two events define convergence to a zero of f, in which case $z^{(k+1)}$ is stored and the number of zeros left to be found is decreased by 1. In the latter two events, the iteration fails to converge. In this case, the iteration is interrupted and restarted with a new randomly chosen initial value $z^{(0)}$.

To avoid repeated convergence to already found zeros, each zero must be removed from the function as it is found, by collocation of a simple pole. For instance, suppose f has two simple zeros $z^{(0)}$ and $z^{(1)}$ in the region of interest. A refining iteration will find one of the zeros, say $z^{(0)}$. Define a new function $f^{(1)}$ which does not have a zero at $z^{(0)}$, but does have a zero at $z^{(1)}$:

$$f^{(1)}(z) = \frac{f(z)}{z - z^{(0)}}. \tag{13}$$

A refining iteration on $f^{(1)}$ will then find the remaining zero $z^{(1)}$. If f has a single zero $z^{(0)}$ with multiplicity 2 the iteration on f will converge to $z^{(0)}$ and so will the iteration on $f^{(1)}$, indicating that $z^{(0)}$ is a double zero. In general, if f has N zeros within the bounded region, the refining procedure is repeated N times with the zero $z^{(\nu)}$ being found from the function

$$f^{(\nu)}(z) = \frac{f(z)}{(z - z^{(0)}) \cdots (z - z^{(\nu-1)})}, \quad \nu = 1, \ldots, N. \tag{14}$$

Systems of functions of several variables

Without loss of generality, consider once again a system of two functions $f_1(z_1, z_2)$ and $f_2(z_1, z_2)$. Suppose, as before, there are p values of z_1 and q values of z_2 satisfying $f_1 = 0$ and $f_2 = 0$ in a bounded region $\mathcal{D}_1 \times \mathcal{D}_2 \subset \mathbb{C}^2$. The total number of zeros of the system is $N = p \times q$. The zeros of the system can be found by applying the refining procedure to $f_2\big(z_1^{(\nu)}, z_2\big)$ as a function of z_2, while at each iteration step finding the p values $z_1^{(\nu)}$ that satisfy $f_1\big(z_1^{(\nu)}, z_2\big) = 0$, with z_2 taken as a parameter. Once the iteration on f_2 converges, a set of p zeros of the system has been found.

The remainder of the refining procedure is identical to the single variable case. For each of the q zeros, an initial value for the iteration is chosen at random within $\mathcal{D}_2$. If the iteration jumps out of $\mathcal{D}_2$ or the number of iterations exceeds a specified maximum, the procedure is interrupted and restarted with a new initial value in $\mathcal{D}_2$. If the iteration converges, the set of zeros $\big\{(z_1^{(0)}, z_2^{(k)}), \ldots, (z_1^{(p)}, z_2^{(k)})\big\}$ is stored. As for the single variable case, the function f_2 is then redefined in order to remove the zero $z_2^{(k)}$ just found. This procedure is repeated until all q zeros $z_2^{(k)}$ are found.

EXAMPLE APPLICATIONS IN DUCT ACOUSTICS

The 2D lined duct with flow

A 2D uniform duct extends from $-\infty$ to $+\infty$ in the direction of the z coordinate, and is bounded by locally reactive walls at $y = 0$ and $y = 2h$ (figure 2). The medium in the duct is assumed to flow steadily with a uniform profile and Mach number M. The problem admits separated solutions of the form $p(y, z, \omega) = Y(y)Z(z)e^{i\omega t}$ which, for duct walls with equal specific admittance $A_1 = A_2 = A$, and even modes, lead to the eigenvalue equation

$$k_y h \sin(k_y h) - ikhA\left(1 - M\frac{k_z^{\pm}}{k}\right)^2 \cos(k_y h) = 0, \tag{15}$$

where the axial wavenumber $k_z^{\pm}$ is related to the transverse wavenumber k_y through the dispersion relation

$$\frac{k_z^{\pm}}{k} = \frac{-M \pm \sqrt{1 - (1 - M^2)\left(\frac{k_y}{k}\right)^2}}{1 - M^2}. \tag{16}$$

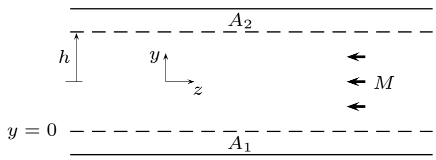

Figure 2: Schematic of the two dimensional duct with half-height h, flow Mach number M, and locally reacting walls with acoustic admittance A_1 and A_2.

The left hand side of equation (15) defines the complex function $f(k_y)$, and its zeros are the eigenvalues associated with the modes of even symmetry. Figure 3 reproduces Table 1 of ref. [2] (see also [3]) in graphical form. It shows the zeros of $f(k_y)$ found within the rectangle with top right corner $30+2i$ and lower left corner $0-6i$, superimposed on a contour level plot of $\log\big(|f|\big)$. The parameters in the calculation were $M = -0.5$, $kh = 1$ and $A = 0.72 + 0.42i$. The contour levels show the approximate location of the zeros.

The plots show how all eigenvalues can be found when there are branch points within the region of interest. The original rectangular region was split at the branch point, marked with the symbol $\square$, and the branches of the square root in equation (16) were defined such that the branch cuts radiate away from each sub-region of interest. The eigenvalues found in this way are all the eigenvalues in the given region extended to the two sheets of the Riemann surface. The topmost plots show the eigenvalues associated with a positive sense of propagation, whereas the bottom plots show the eigenvalues for negative propagation. Note in the bottom right plot the eigenvalue with a relatively large imaginary part, this is associated with a mode of the surface wave type.

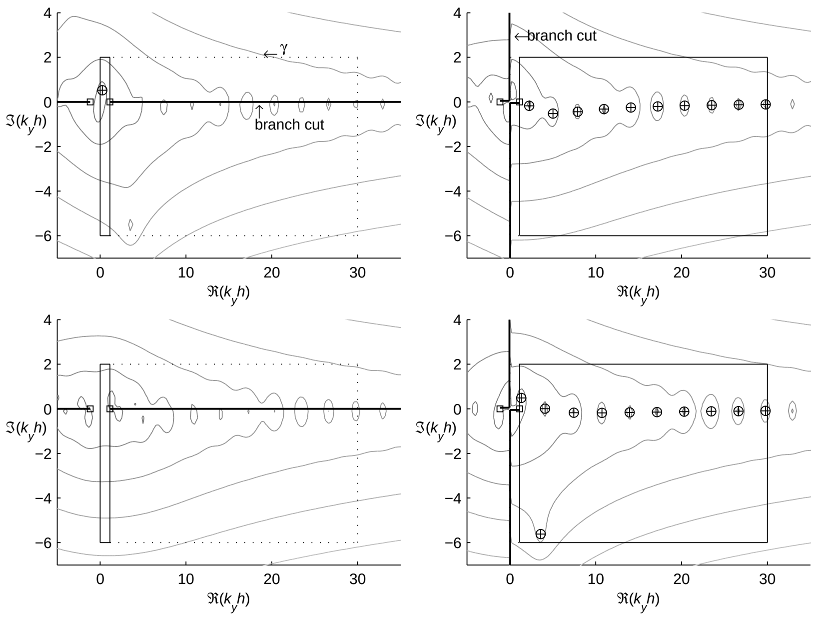

Figure 3: Contour level plot of $\log\big(|f(k_y)|\big)$ showing the approximate locations of the eigenvalues corresponding to even modes, for the parameters $M = -0.5$, $kh = 1$ and $A = 0.72 + 0.42i$. The eigenvalues found within each sub-region are shown by the symbol $\oplus$.

The lined rectangular duct with flow

Coupled sets of eigenvalue equations can occur in a variety of problems. A classic example is the problem of the infinite duct of rectangular cross-section, with uniform flow and finite admittance walls (figure 4). Analytic modal solutions may be found for this problem but the equations for the transverse

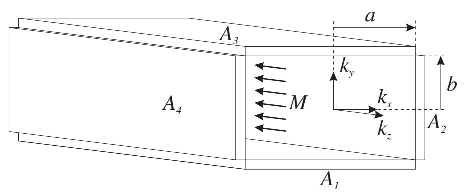

Figure 4: Geometry schematic for the rectangular-section duct with flow and different acoustic admittance on each wall.

eigenvalues become coupled in the presence of flow, through the dispersion relation, and must be solved simultaneously. The problem is treated in ref. [5]. For $A_1 = A_3 = A_y$ and $A_2 = A_4 = A_x$, and for even modes in both transverse directions, the eigenvalue equations are

$$\begin{cases} k_x a \sin(k_x a) - ikaA_x \left(1 - M\dfrac{k_z^{\pm}}{k}\right)^2 \cos(k_x a) = 0 \\ k_y b \sin(k_y b) - ikbA_y \left(1 - M\dfrac{k_z^{\pm}}{k}\right)^2 \cos(k_y b) = 0 \end{cases}, \tag{17}$$

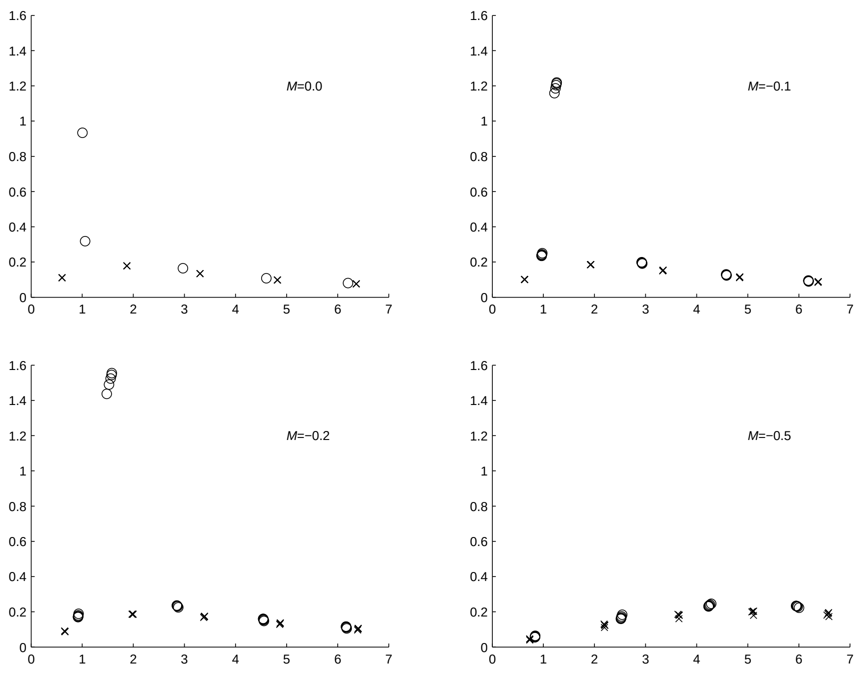

Figure 5: Transverse eigenvalues associated with even modes in both directions for the lined rectangular duct, for different values of the flow Mach number M. $ka = kb = 10$, $A_x = 0.1{+}0.1i$, and $A_y = 0.1{-}0.1i$. $\circ$ k_x, $\times$ k_y.

where

$$k_z^{\pm}/k = \left[-M \pm \sqrt{1-(1-M^2)(k_x^2+k_y^2)/k^2}\right]/(1-M^2). \tag{18}$$

Upon eliminating $k_z^{\pm}$ by substituting equation (18) in equations (17), the left hand side of the resulting equations defines a system of two complex functions of two complex variables $f_1(k_x, k_y)$ and $f_2(k_x, k_y)$. As before, the problem of finding the eigenvalues reverts to finding the zeros of this system. Figure 5 shows the eigenvalues found with the coupled solver in the rectangular region with top right corner at $7+10i$ and left bottom corner at $0-10i$, for positive propagation and even modes in both transverse directions, $ka = kb = 10$, $A_x = 0.1+0.1i$, and $A_y = 0.1-0.1i$. The eigenvalues k_x are shown by the symbol $\circ$, and the eigenvalues k_y are shown by the symbol $\times$. The top left plot is for $M = 0$. In this case, there is no coupling between the eigenvalue equations, and each eigenvalue has infinite multiplicity, i.e. every pairing of an eigenvalue k_x with an eigenvalue k_y is a solution of the system. In the given region, there are 5 eigenvalues for each transverse direction, so that there are in total 25 roots of the system (17). The remaining plots show the eigenvalues found within the same region for $M = -0.1$, $M = -0.2$, and $M = -0.5$. Note that, in order to maintain comparable scales, the eigenvalues associated with the modes of surface wave type are not shown for the latter case, although they were found by the solver. The effect of the coupling is well visible and becomes stronger as the Mach number increases. Where there was one eigenvalue with higher multiplicity, there is now a cluster of eigenvalues with mulitiplicity 1. Within the region of interest, there are 5 clusters of 5 eigenvalues for each transverse direction, again totaling 25 pairs satisfying the system (17). Each eigenvalue in these pairs is now unique.

SUMMARY

In order to express the acoustic field in lined ducts in terms of modes, complex transcendental equations most be solved. The solutions(s) can be found numerically by locating zeros of complex functions. The Argument Principle provides an effective tool for counting zeros of analytic functions in a bounded region of the complex plane, and is applicable to systems of complex functions in multi-dimensional domains. The combination of a procedure for counting zeros with a global Newton-Raphson method provides a reliable eigenvalue solver for systems of coupled eigenvalue equations, guaranteed to find all eigenvalues in a given bounded region. The solver is particularly suited for use in large scale optimization studies, where there is a need for a robust, reliable, and efficient algorithm for finding zeros of multi-dimensional complex functions. In addition to these qualities, the algorithm described in this paper benefits from being simple and easy to implement.

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