SOUND SCATTERING FROM THE EDGE OF A SUBMERGED COATED WEDGE

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Abstract.
We developed a mathematical approach for predicting acoustic scattering from a wedge for various boundary conditions and wedge material properties. Our approach includes wedge boundary conditions that can be characterized by an impedance, including absorption. Numerical predictions are found using the Kontorovich-Lebedev transform [A. Erdelyi, *Higher Transcendental Functions* (McGraw-Hill, 1953)] based upon MacDonald functions, requiring very high precision numerical techniques. We apply this numerical approach to calculate the scattering amplitude for a fabricated wedge structure that has been measured in non-laboratory environment.

INTRODUCTION

Solutions for the acoustic wedge problem that can be found in the early literature concern exclusively the case of an impenetrable wedge. These analytical results fall roughly into three classes:

(1) Eigenfunction expansions (e.g., Ref. [1]).
(2) Integral expressions based on Sommerfeld's work, see e.g., Ref. [2].
(3) Direct solutions of the wave equation with subsequent development of asymptotic expansions [3] which is of interest here.

We shall be interested mainly in the case of distant sources (i.e. incident plane waves). The plane-wave case can be considered a special case of point sources being moved to infinity. This will, however, only lead to plane waves incident normally onto the wedge's edge, oblique incidence having been treated by Carslaw [2] only. We shall base our following considerations on the developments of type (3). It will then be necessary for us to assume point or line sources and to subsequently specialize to normally-incident plane waves. The theory of Ref. [3] will be generalized to a wedge with impedance boundary conditions, and the surface impedance corresponding to a lossy acoustical coating will be obtained.

SOLUTION OF THE WEDGE PROBLEM FOR AN IMPEDANCE BOUNDARY CONDITION

In this section, we generalized the solution which was found by Oberhettinger [3] for the scattering of a sound wave from the edge of a perfectly reflecting wedge, to the case that the field does not satisfy a Dirichlet or Neumann boundary condition on the wedge surface, but an impedance boundary condition. Such a condition is capable of describing the effect of a lossy coating on the wedge surface. The acoustic impedance obtained in a later section of this study.

First, we consider a line source $Q(\rho'\varphi')$ parallel to the edge of the wedge; we use cylindrical coordinates (ρ,φ,z) where the z-axis coincides with the edge. Observation of the field takes place at $P(\rho,\varphi)$ and the geometry is assumed for a wedge with opening angle $\alpha>\pi/2$, i.e. for a wedge in the true sense. The acoustic field is assumed with time dependence

$$\Phi = U(\rho,\varphi)\exp(i\omega t) \tag{1}$$

where U satisfies the two-dimensional wave equation $\Delta^2U + k^2U = 0$. This equation is solved by outgoing waves of the reflected field,

$$U = \exp(\pm i\mu\varphi)H_\mu^{(2)}(k\rho) \tag{2}$$

containing the Hankel function with index μ left open. Oberhettinger [3] chooses to temporarily convert this wave problem into an exponential-decay problem by setting $k \equiv 2\mu/\lambda \equiv \omega/c = -i\gamma$ with corresponding change $\Phi = U(\rho,\varphi)\exp(\gamma ct)$, $\Delta^2U - \gamma^2U = 0$, and a corresponding solution

$$U = \exp(\pm i\mu\varphi)K_\mu(\gamma\rho) \tag{3}$$

containing the modified Hankel function (or MacDonald function)

$$K_\mu(z) = -(i\pi/2)\exp(-i\mu\pi/2)H_\mu^{(2)}(-iz). \tag{4}$$

The wave problem solution is obtained by substituting $\gamma = ik$ at the end, and convergence problems are avoided in this way.

The incident field, emitted by the line source $Q(\rho'\varphi')$ and observed e.g at point $P(\rho,\varphi)$, is given by

$$U_i = H_0^{(2)}(kr) = (2i/\pi)K_0(\gamma r) = (2i/\pi)K_0(\gamma[\rho^2 + \rho'^2-2\rho\rho'\cos(\varphi-\varphi')]^{1/2}. \tag{5}$$

The total field U consists of the incident field U_i plus a reflected field U_r that has no singularity within the wedge, $U = U_i + U_r$. This total field has to satisfy the appropriate boundary conditions on the wedge surfaces. In our case, this is the impedance boundary condition. Acoustic impedance Z of the surface is defined as the ratio $Z = p/v_n$, where p is the acoustic pressure and v_n the normal component of the particle velocity pointing into the impedance medium. We may consider our solution Φ to represent the velocity potential of the field, from which p and v can be derived as

$$P = \rho_0 \partial\Phi/\partial t = i\rho_0\omega\Phi = i\rho_0\omega U \exp(i\omega t) = \rho_o c\gamma U \exp(\gamma ct) \qquad (6)$$

where ρ_0 is the density of the ambient acoustic medium, and

$$\mathbf{v} = -\mathrm{grad}\Phi = -\mathrm{grad}U\exp(i\omega t) = -\mathrm{grad}U\exp(\gamma ct). \qquad (7)$$

Accordingly, $\rho_0 c\gamma U = \pm Z\partial U/\rho\partial\varphi$ will be our boundary condition on the walls $\varphi = \pi$ and $\varphi = \alpha$ of the wedge, respectively, with Z a given function determined by the acoustic properties of the wedge surfaces. For a wedge that is coated by a lossy material, an expression for Z will be obtained in a later portion of this study.

In order to facilitate satisfying the boundary conditions for our field $U = U_i + U_r$ it will be expedient to introduce integral representations for U_i and U_r as given by Oberhettinger [3] (representing a Kontorovich-Lebedev transform):

$$U_i = (4i/\pi^2)\int^{\infty}_0 K_{i\nu}(\rho\varphi)(\gamma\rho')\cosh(\nu(\pi-|\varphi-\varphi'|))d\nu \qquad (8)$$

which, with $\mu = i\nu$, has the desired form of Eq. (3), and

$$U_r = (4i/\pi^2)\int^{\infty}_0 K_{i\nu}(\gamma\rho)K_{i\nu}(\gamma\rho')[f_1(\nu)\exp(\nu\varphi) + f_2(\nu)\exp(-\nu\varphi)]d\nu, \qquad (9)$$

which is also of this form and which contains the unknown functions f_1 and f_2 to be determined by the boundary conditions.

At $\varphi = 0$, the boundary condition leads to $f_1 + f_2 + C_0 = \beta(f_1 - f_2 + S_0)$, where we have introduced the notations $C_0 = \cosh(\nu(\pi-\varphi'))$, $S_0 = \sinh(\nu(\pi-\varphi'))$ and $\beta = \zeta\nu/\gamma\rho$, where $\zeta = Z/\rho_0 c$ is called the specific acoustic impedance of the surface. Using the notation $C_\alpha = \cosh(\nu(\pi-\alpha + \varphi'))$, $S_\alpha = \sinh(\nu(\pi-\alpha + \varphi'))$, these equations are solved for f_1 and f_2 to give

$$f_1(\nu) = (1/2)[(1-\beta)(C_o - \beta S_o)e^{\nu\alpha}-(1 + \beta)(C_\alpha- \beta S_\alpha)]/[2\beta\cosh\nu\alpha + (1+\beta^2)\sinh\nu\alpha], \qquad (10a)$$

$$f_2(\nu)=-(1/2)[(1 + \beta)(C_o - \beta S_o)e^{\nu\alpha}-(1-\beta)(C_\alpha-\beta S_\alpha)]/[2\beta\cosh\nu\alpha + (1 + \beta^2)\sinh\nu\alpha]. \qquad (10b)$$

These expressions, inserted into Eq. (9), then lead to the explicit solution for the reflected velocity potential U_r given by Eq. (9) in integral form; for Dirichlet or Neumann boundary conditions, the total solution for $U = U_i + U_r$ would then correspond to Oberhettinger's Equation (14).

It is quite possible that the wave problem solution can be directly obtained out of the exponential-decay problem solution (substituting $\gamma = ik$) with a convergent integral if the "principle of minimal absorption" is employed (since in practice the ambient medium in which the acoustic wave propagates, is ever so slightly absorptive). This corresponds to replacing k in the substitution $\gamma = ik$ by $k + i\varepsilon$, $\varepsilon>0$ which renders the integral convergent at its upper limit (and then letting $\varepsilon \rightarrow 0$). We shall show below that numerically convergent forms of integration can be found in this way.

This latter development was done for a plane-wave incident field, which is probably for us the most interesting case. The way this was carried out by Oberhettinger is by first transforming the line-source case into the point-source case by an additional integration, Eq. (26) of Ref. [3], and then letting the point source position tend to infinity. We here propose to obtain our incident plane-wave solution from the line-source solution, Eq. (9), or from corresponding equations of our subsequent developments, by letting our line source tend to infinity, and, of course, reverting from the exponential-decay case to the wave case by replacing γ by ik.

The results of our numerical evaluation of the integrals in the scattering solution will be presented in a subsequent section of our study, see below. At this point, however, we shall proceed to the other topic of the investigation, namely to derive expressions for the surface impedance that apply when physical boundary planes are covered by a lossy acoustic coating.

OBTAINING THE SURFACE IMPEDANCE OF A COATED PLANE

The surface impedance will be found for the case of an absorbing solid layer bounded by a fluid (water) on one side, and by a metal substrate on the other, see Fig. 1. If the absorbing layer on top of the metal has sufficiently high absorptivity, it can be treated as a semi-infinite medium and the presence and configuration of the metal substrate will not have any relevance. The surface impedance Z that enters our solution of Eqs. (9) and (10), through the parameters β and ζ will thus be calculated on that basis. This calculation can follow the procedure taken by Brekhovskikh [4]. The geometry is shown in Fig. 1. A wave $\mathbf{k}_1$ is incident through the fluid onto the fluid-solid boundary under the angle θ_1, and reflected back into the fluid under the same angle. In the solid, there is a compressional wave $\mathbf{k}_2$ emitted under the angle θ_2, and a shear SV wave (vertical polarization) $\boldsymbol{\kappa}_2$ emitted under the angle γ_2. No SH wave is created in the solid since an incident compressional wave whose particle motion takes place in the xz plane cannot accomplish this.

The angles are governed by Snell's law which guarantees that the horizontal wave vector components ξ are the same in both media: $\xi = k_1 \sin\theta_1 = k_2 \sin\theta_2 = \kappa_1 \sin\gamma_2$, allowing the satisfaction of the boundary conditions independently of the values of x. The boundary conditions are, explicitly, the continuity of the stress tensor components σ_{zz} and σ_{xz} in the z direction, as well as of the z component of the particle displacements u_z (no continuity of u_x applies since the fluid may slip freely along the boundary).

Following Brekhovskikh's procedure, the result will furnish for us the overall surface impedance Z needed in our wedge problem, i.e. in the eqs. for U_r, β, ζ, f_1 and f_2. To this end, we relate the quantities p and v_n in our expression for Z. This gives the desired surface impedance of the coating as

$$Z = Z_o(1 + V)/(1-V), \tag{11}$$

expressed by the individual impedances $Z_o = \rho_1 c_1/\cos\theta_1$, $Z_L = \varphi_2 c_2/\cos\theta_2$, $Z_T = \zeta_2 b_2/\cos\gamma_2$ with b_2 the shear speed in the coating, and the reflection coefficient

$$V = (Z_L\cos^2 2\gamma_2 + Z_T\sin 2\gamma_2 - Z_o)(Z_L\cos^2 2\gamma_2 + Z_T \sin^2 2\gamma_2 + Z_o). \qquad (12)$$

The result, **not** found in Brekhovskikh's book, constitutes a very attractive expression for Z, i.e. as the intrinsic surface impedance $Z_o = \rho_1 c_1/\cos\theta_1$ of the fluid modified by the reflection coefficient (V) from the solid-coating half space.

Since the case may arise where the absorbing solid coating layer is thin enough so that the effect of a substrate beneath the layer becomes noticeable, we here obtain the surface impedance on the fluid-coating interface this situation arises the surface impedance on the fluid-coating interface for this situation also. (The above derivation has assumed that the layer is of semi-infinite thickness).

A study for a related situation was carried out by Glegg [5], who assumed a fluid column of height h, density ρ_1 and sound speed c_1 overlying a solid sediment layer of thickness d, density ρ_2 and compressional (shear) speed $c_2(b_2)$ which in turn is lying on top of a semi-infinite substrate of density ρ_3 and compressional (shear) speed $c_3(b_3)$. His results can be used for our case, including a metal substrate with appropriate parameters.

Using the corresponding developments furnishes an explicit expression for Z to be utilized in Eqs. (10a, b) in order to obtain the scattered field, Eq. (9), from a two-layered impedance covered wedge.

NUMERICAL RESULTS

This section will present some examples of calculated values of surface impedance, and of scattering strength based on the above-presented theories. Figure 2 plots vs. frequency the dilatational impedance of the commercially available coating material Rubatex, manufactured by RBX Corporation, Roanoke, VA. Figure 3 shows the 2-layer effective impedance of a ½″ steel plate coated with 1" of Rubatex at 1kHz, vs. incident angle from the normal. It is found that with a doubling of the Rubatex coating, a substantially reduced effective surface impedance is achieved.

As to scattering from a wedge, Figure 4 shows the target strength of an impedance wedge of opening angle 10^o. Plots are for coating material when only the real or the imaginary parts are considered alone, and where the complete complex impedance is considered, being $Z = 5.4 \times 10^5(1 + 0.01i)$. The incident wave propagates along the bisector of the wedge, and the scattered wave is observed at 1 km distance on the bisector. Some results for a coated wedge were also obtained by Norris and Osipov [6], using a theory by Malyuzhinets.

SUMMARY

This study considers wedges coated with absorbing material of one or two layers. The absorptivity of layered coatings is studied as a function of layering , using theories of Brekhovskikh or Glegg. A coating material of a given impedance (describing its absorptivity) is considered to be attached to the outside of a steel wedge, and the target strength of such a coated wedge is obtained using our mathematical approach based on the theory of Oberhettinger.

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