ICSV13 - Vienna
The Thirteenth International Congress on Sound and Vibration
Vienna, Austria, July 2-6, 2006

A MIDDLE FREQUENCY METHOD BASED ON SPACE-AVERAGED QUADRATIC VARIABLES FOR ONE-DIMENSIONAL SYSTEMS

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Abstract

This paper presents a method based on quadratic variables, some of which keep a strong energy meaning, even when time- and space-averaged. If the damping is assumed to be slight, the governing equation of this method is similar with equations of the Vibrational Conductivity Approach (VCA).

The main point of this work is the presentation of appropriate boundary conditions for space-averaged quadratic variables. The numerical example given proves the space-averaged quadratic method is well suited to model global energy transfers along one-dimensional dissipative structures in the middle frequency range. The simplification of space-averaged quadratic fields reduces discretization requirements, but the lack of spatial information of SEA results is overcome.

INTRODUCTION

Vibration and noise predictive tools are necessary to define noise paths and implement efficient control strategies. Unfortunately the prediction of the vibration behavior of structures consisting of numerous connected elements cannot be correctly done throughout the entire audible frequency range by a single predictive tool. Element Methods (FEM and BEM) are effective only for low frequencies since in the high frequency range the short wavelength of the system deformation can require the use of an excessive number of degrees of freedom and can also make the system very sensitive to manufacturing imperfections [1]. Statistical Energy Analysis (SEA) can give a space and frequency determination of quadratic vibro-acoustical response to the dynamic problem at high frequencies for complex buil-up systems, but the resulting information only consists in averaged quantities describing the behavior on a population of modes of the subsystems. Moreover SEA relies on an energy diffusion model which is not always suitable for multi-mode wave fields with low modal density.

The modern approach of high frequencies problems is based on the energy flow. Its motivation is to overcome some limitations of SEA, especially the lack of spatial information of its results. With this approach, the high frequency acoustic energy flow or the structural intensity of independent plane waves is assumed to be proportional to the gradient of energy density. This leads to a partial differential equation of the heat conduction type, which is the basis of the Vibrational Conductivity Approach (VCA) [2]. Although the basic VCA equation was applied with success to one-dimensional structures [3], the justification of the VCA is more difficult and its validity is not satisfying for multi-dimensional problems [4].

This present work focuses on the concept of space-averaged quadratic variables as other works did before [5, 6]. Here quadratic variables averages are over time and space for steady state harmonic waves.

SPACE-AVERAGED QUADRATIC VARIABLES

General Form of the Displacement Field and Energy Quantities

Systems considered in this work are one-dimensional systems in which only longitudinal waves propagate. So the method presented here is applicable to acoustics by using the speed of sound c_0 to link the wavenumber k to the angular frequency ω: $k = \omega/c_0$. Only general and non restrictive assumptions are put forward: **i)** small displacement and small strain, **ii)** homogeneous and isotropic medium with hysteretic damping (density ρ, damping factor η, complex Lamé coefficients $\lambda = \lambda_0(1 + j\eta)$ and $\mu = \mu_0(1 + j\eta)$), **iii)** steady state harmonic waves of angular frequency $\omega = 2\pi f$, **iv)** system with large dimensions compared to the wavelength λ_0.

In such a system (and out from exciting sources), the scalar potential ϕ and the displacement $\boldsymbol{u} = \boldsymbol{grad}\ \phi$ satisfy the scalar and the vectorial Helmholtz equation, respectively $(\Delta + k^2)\phi = 0$ and $(\Delta + k^2)\boldsymbol{u} = 0$, where the wavenumber is $k = \sqrt{\rho\omega^2/(\lambda + 2\mu)}$. So the complex amplitude $\hat{u}$ of the displacement field between two junctions writes $\hat{u}(x) = A\ e^{-jkx} + B\ e^{jkx}$, where A and B are the complex amplidudes, respectively of the forward and the backward propagating wave. Writing the wavenumber $k = k_0(1 - j\theta)$ leads to $k_0 = \sqrt{\rho\omega^2\eta^2/(2(\lambda_0 + 2\mu_0)(1 + \eta^2)(\sqrt{1 + \eta^2} - 1))}$ and $\theta = (\sqrt{1 + \eta^2} - 1)/\eta$. Among quadratic variables are the kinetic energy density T, the strain energy density U, the total energy density $W = T + U$, the lagrangian energy density $L = T - U$ and the structural intensity $\boldsymbol{I}$. Their complex amplitudes (the notation $\hat{}$ refers to and whose real parts stand for time averages) are obtained from the displacement field. This is detailed in Table 1, where $*$ refers to the conjugate number. For example the complex strain energy density $\hat{U}$ writes:

structural intensity	$\hat{I} = j\omega(\lambda + 2\mu)u_{,x}u^*/2$
kinetic energy density	$\hat{T} = \rho\omega^2 uu^*/4$
strain energy density	$\hat{U} = (\lambda + 2\mu)u_{,x}u^*_{,x}/4$
total energy density	$\hat{W} = \hat{T} + \hat{U}$
Lagrangian density	$\hat{L} = \hat{T} - \hat{U}$

Table 1: *Complex amplitudes of quadratic variables.*

$$\hat{U}(x) = \frac{\lambda + 2\mu}{4}\ kk^*\ \left(AA^*\ e^{-j(k-k^*)x} + BB^*\ e^{j(k-k^*)x} + A^*B\ e^{-j(k+k^*)x} + AB^*\ e^{j(k+k^*)x}\right). \tag{1}$$

The kinetic energy density $\hat{T}$ is an exception as the only purely real quadratic quantity among those presented in Table 1. Time averages of quadratic variables are governed by two wavenumbers $k + k^*$ and $k - k^*$ matching with two different scales of variation (Figure 1), and $k - k^*$ is the wavenumber that describes power transfers at a large scale compared to the wavelength [8].

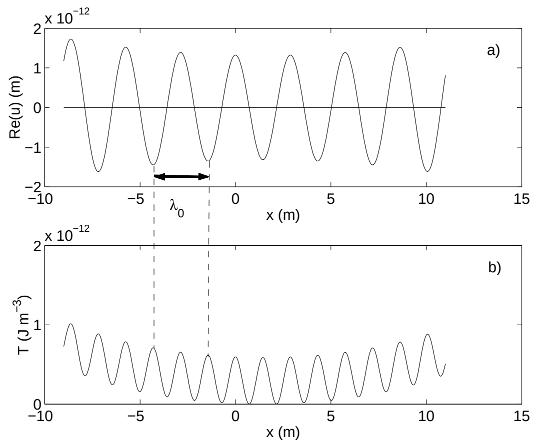

Figure 1: *One-dimensional longitudinal counter-propagative waves out from exciting sources in a steel medium (medium 1 in Table 2 except damping factor $\eta = 0.07$) at $2100Hz$: a) Real part of the displacement u ; b) kinetic energy density T.*

Space-Averaged Variables and Large Scale Components for 1D Plane Waves

At the scale of the half wavelength $\lambda_0/2 = 2\pi/(k + k^*)$, the space average of any quadratic variable $< Q >$ is defined as following:

$$< Q > (x) = \frac{2}{\lambda_0} \int_{x-\lambda_0/4}^{x+\lambda_0/4} Q(u)du, \tag{2}$$

where $\lambda_0/2$ also is the scale of variation of small scale components of Q (Figure 1). Large scale components of quadratic variables, with wavenumber $k - k^*$ appearing in (1), are linked to their half-wavelength-scale space-averaged values defined by (2). For example the space-averaged strain energy density $<\hat{U}>$ writes

$$<\hat{U}>(x)=\frac{\lambda+2\mu}{4}\ kk^*\ \left(AA^*\ e^{-j(k-k^*)x}+BB^*\ e^{j(k-k^*)x}\right)\ f(\theta). \tag{3}$$

The factor $(\lambda+2\mu)\ kk^*\ \left(AA^*\ e^{-j(k-k^*)x}+BB^*\ e^{j(k-k^*)x}\right)/4$ stands for the large scale components of $\hat{U}$, which are obtained by neglecting interferences between the two considered plane waves. The averaging process also introduces the last factor of equation (3) $f(\theta)=sh(\pi\ \theta)/(\pi\ \theta)$ that depends on the only dissipative properties of the medium. The same process can be used to obtain $<\hat{T}>$, which is found to be proportional to $<\hat{U}>$:

$$<\hat{U}>=<\hat{T}>(1+j\theta)/(1-j\theta). \tag{4}$$

Therefore the result is similar for the other space-averaged energy densities $<\hat{W}>=<\hat{T}>+<\hat{U}>$ and $<\hat{L}>=<\hat{T}>-<\hat{U}>$, and also for the space-averaged structural intensity $<\hat{I}>$.

Differential Equation for the Space-Averaged Structural Intensity

Only the propagation of large scale components of quadratic variables is considered here. Between junctions, the space-averaged structural intensity $<I>$ satisfies the following differential equation:

$$\Delta<I>+(k-k^*)^2<I>=0. \tag{5}$$

Space-averaged energy densities $<T>$, $<U>$, $<W>$ and $<L>$ also satisfy this differential equation. Since the wavenumber $k-k^*$ is purely imaginary, the space-averaged structural intensity $<I>$ is the evanescent solution of a propagation equation. Equation (5) is available for any value of the damping loss factor η, but in the case of a slight damping, it writes the same as the basic VCA equation previously derived in many works, originally in [2] and for example in the General Energy Method (GEM) [3] for beams and rods.

Space-Averaged Energy Densities

Here the choice has been made to work with the space-averaged structural intensity $<I>$ and other space-averaged quadratic variables $<T>$, $<U>$, $<W>$ and $<L>$ can be derived from $<I>$. Similarly to acoustics [7], the structural intensity and energy densities (Table 1) are linked by:

$$I_{,x}=-2j\omega(T-U)+P, \tag{6}$$

where the input power density P, associated with the external load $\boldsymbol{f_x}$, is defined as

$$P=-j\omega\boldsymbol{f_x}.\boldsymbol{u^*}/2. \tag{7}$$

Averaging equation (6) and using the proportionnality (4) between $<T>$ and $<U>$ leads to $<T>$ for example. Then other averaged energy densities $<U>$ (4), $<W>$ and $<L>$ are derived from $<T>$.

BOUNDARY CONDITIONS

Space-averaged quadratic quantities, first $< I >$ and then $< T >$, $< U >$, $< W >$ and $< L >$, are obtained by solving the local equation (5) with appropriate boundary conditions for either passive or active junctions and which are presented in this section.

Active Junctions

At $x = a$ where the concentrated load is f_x, $< I >$ and $< I >_{,x}$ are discontinuous. The input power density P (7) is an appropriate parameter expected to appear in boundary conditions for active junctions. The two conditions written here are about the left limit value $< I > (a^-)$ and the right limit value $< I > (a^+)$: the active junction is qualified by $< I > (a^+) - < I > (a^-)$ and $< I > (a^+)/ < I > (a^-)$ [8]. The first boundary condition writes

$$< I > (a^+) - < I > (a^-) = \frac{P}{2}\left(1 + \frac{k^*}{k}\frac{P^*}{P}\right) f(\theta) \tag{8}$$

whereas the second one introduces the brought back impedance Z_{bb} from $x = -L_1$ back to $x = a^-$

$$Z_{bb} = \rho c\, \frac{e^{jk(L_1+a)} - e^{-jk(L_1+a)} + z_1\big(e^{jk(L_1+a)} + e^{-jk(L_1+a)}\big)}{z_1\big(e^{jk(L_1+a)} - e^{-jk(L_1+a)}\big) - e^{jk(L_1+a)} - e^{-jk(L_1+a)}}. \tag{9}$$

and finally writes

$$\frac{< I > (a^+)}{< I > (a^-)} = 1 + \frac{P\left(1 + \dfrac{k^* P^*}{kP}\right)}{2j\dfrac{PP^*}{f_x f_x^*}\left(Z_{bb} - \dfrac{k^* Z_{bb}^*}{k}\right)}. \tag{10}$$

Passive Junctions

Impedance Condition

The impedance condition about the complex amplitude of the displacement u writes $u_{,n}(L) + jkzu(L) = 0$, where z is the specific impedance at $x = L$: $z = z_1$ when $L = -L_1$ and $z = z_2$ when $L = L_2$. Using this condition and the displacement form finally leads to the following mixed boundary condition for the complex amplitude of the space-averaged structural intensity [8]:

$$< I >_{,n} (L) + j(k - k^*)\frac{1 + zz^*}{z + z^*} < I > (L) = 0. \tag{11}$$

Discontinuity of the Material Density

Boundary conditions for a discontinuity of the material density at $x = d$ (Figure 2) are explained here. The three specific impedances z_{a^-}, z_a, z_{a+} and the three specific impedances z_{b^-}, z_b, z_{b+} are respectively linked by $z_{a^+} + z_{a^-} = z_a$ and $z_{b^+} + z_{b^-} = z_b$. At the boundary $x = d$ between the two media the displacement and the normal stress are continuous: $u(d^-) = u(d^+)$ and $(\lambda_1 + 2\mu_1)u_{,x}(d^-) = (\lambda_2 + 2\mu_2)u_{,x}(d^+)$.
At $x = d$ the space-averaged structural intensity is continuous. So the first boundary condition writes

$$< I > (d^+) = < I > (d^-). \tag{12}$$

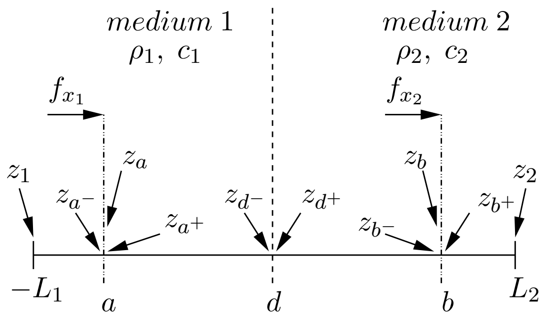

Figure 2: *Discontinuity of the material density: configuration of the studied system.*

Since specific impedances z_{d^-} and z_{d^+} can be computed from specific impedances z_1 and z_2 as brought back impedances, mixed boundary conditions in the form (11) can be written for space-averaged structural intensities $< I > (d^-)$ and $< I > (d^+)$. These lead to the second boundary condition about the ratio $< I >_{,x} (d^+)/ < I >_{,x} (d^-)$, which finally writes:

$$\frac{< I >_{,x} (d^+)}{< I >_{,x} (d^-)} = -\frac{k_2 - k_2^*}{k_1 - k_1^*} \frac{1 + z_{d^+} z_{d^+}^*}{1 + z_{d^-} z_{d^-}^*} \frac{z_{d^-} + z_{d^-}^*}{z_{d^+} + z_{d^+}^*} \tag{13}$$

NUMERICAL EXAMPLE

The approach consists in solving equation (5) along the one-dimensional structure, with boundary conditions: **i)** equations (8) and (10) for a concentrated load, **ii)** equation (11) for a specific impedance z at the end of the system, **iii)** equations (12) and (13) for a discontinuity of material density. Once the space-averaged structural intensity $< I >$ is obtained, space-averaged energy densities $< T >$, $< U >$, $< W >$ and $< L >$ are derived. Computations were carried out for pure longitudinal waves propagating at a frequency of $4000\ Hz$ either in a steel medium (Table 2) between $-L_1 = -10\ m$ and $d = 3\ m$ or in an aluminium medium (Table 2) between $d = 2\ m$ and $L_2 = 10\ m$ (Figure 2). A first concentrated load $f_{x_1} = 1\ Nm^{-2}$ is located at $a = -2\ m$ and a second one $f_{x_2} = 2\ Nm^{-2}$ is located at $b = 7\ m$. The specific impedance z_1 is $z_1 = 0.05 + 0.01j$ means the junction is highly reflecting and a little dissipative. The specific impedance $z_2 = 0.1$ means the junction at $x = L_2$ is highly reflecting. The discontinuity of the material density at $x = d$ implies a discontinuity of the spatial derivative of both local (computed from the displacement field) and space-averaged structural intensities. The averaged formulation enables a good reconstitution of this latter discontinuity (Figure 3a). With the averaged quadratic formulation, the large scale components of energy quantities are well reconstituted . When space-averaged, energy quantities like energy densities become smoother (Figure 3b and c). A model based on the displacement with 6 finite element nodes per wavelength would require 161 nodes at $4000\ Hz$ whereas very few nodes are necessary on $-L_1 \leq x \leq L_2$ for the space-averaged quadratic model presented in this work. This demonstrates the relevance of this averaged energy method as the frequency increases. Moreover resonances can be observed as the highest and lowest values of the input impedance modulus $|Z_a| = |Z_{a^+} + Z_{a^-}|$ at $x = a$ (where $Z_a = f_{x_1}/(j\omega u(a))$ is computed by using brought back specific impedances z_1 and z_2) are changing over (Figure 3d) inside the frequency range $[2000\ Hz - 6000\ Hz]$.

	medium 1	medium 2
Density $\rho\ (kg\ m^{-3})$	7800	2700
Young Modulus $E\ (Pa)$	$2.1\ 10^{11}(1 + j\ 0.01)$	$0.7\ 10^{11}(1 + j\ 0.01)$
Poisson ratio ν	0.3	0.3

Table 2: Properties of media

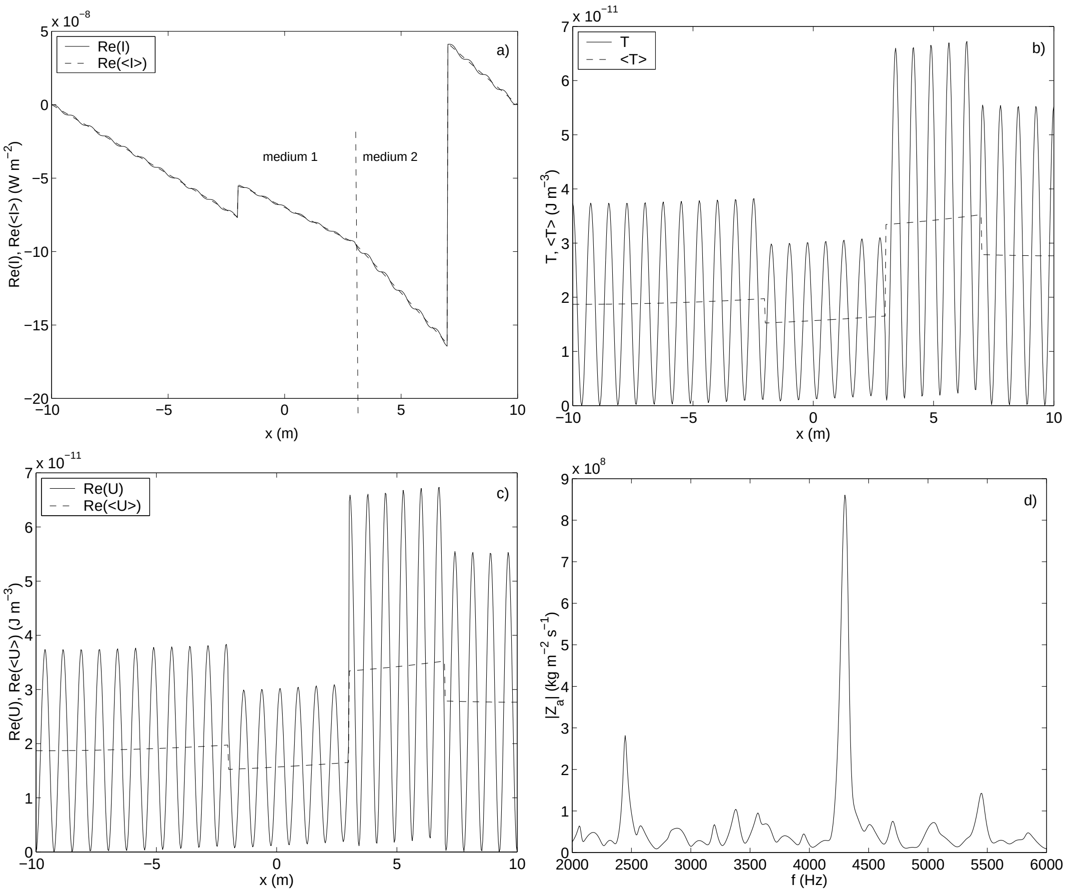

Figure 3: Energy quantities at $4000\ Hz$: a) real part of the structural intensities, b) kinetic energy densities, c) real part of the strain energy densities. Solid line: solution obtained from the displacement field, dashed line: solution obtained from the space-averaged quadratic method ; d) modulus of the input impedance Z_a at $x = a$.

CONCLUSIONS

The link between large scale components of quadratic variables in plane waves and their space-averaged value, when the average is made along a half wavelength, enabled the development of a quadratic formulation based on space-averaged variables to describe one-dimensional power transfers at a large scale compared to the wavelength. Boundary conditions for the space-averaged structural intensity have been obtained from the displacement formulation, in cases of both active and passive junctions. Lastly a numerical example has been given and when compared to the solution of the displacement formulation, it proves this averaged quadratic method is relevant to model global energy transfers along one-dimensional dissipative structures in the middle frequency range: **i)** the space average leads to a simplification of quadratic fields, but the approximation still enables to access the modal behavior and resonances, in particular from the input impedance, **ii)** the resulting quadratic response is more space-detailed than the one of SEA since the space average interval is well defined as the half wavelength instead of SEA subsystems.

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