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ENERGY MEAN AND VARIANCE PREDICTION FROM FREE-INTERFACE SUBSYSTEM MODAL MODELS

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Abstract

This paper concerns the higher frequency vibrations of built-up structures and in particular systems comprising two subsystems connected via an arbitrary interface. The individual subsystems are analyzed using free-interface component mode synthesis. Analytical expressions are developed which represent the dynamic couplings between the sets of subsystem modes. These expressions explicitly include the effects of uncertainty in the parameters of the subsystems, the uncertainty being introduced into the modal properties of the subsystems. The general aims are to predict ensemble mean and variance of the subsystem energy. The dynamic coupling between the two subsystems is seen to fall into three categories: (1) strongly connected, (2) weakly connected and (3) weakly coupled. Effective mass and damping are introduced to account for the dynamic influence on a particular subsystem mode of the remaining modes after the subsystems are coupled. It is seen that for weak connection/coupling, the inter-modal coupling relations reduce to those of two sets of stiffness-coupled oscillators (with modified damping values, if appropriate). As a result, the model can be accommodated within a recently developed theory for the prediction of the ensemble mean and variance of the energy response of coupled oscillator sets. This provides an approach to the analysis of complex, built-up systems in a more systematic way than conventional SEA. Numerical examples are given for two plates connected by a single point. The present theory is seen to give good predictions for weak connection/coupling when compared with Monte Carlo simulations, if the statistical overlap of the system is sufficiently large. The approach helps to give insight into the physics of the coupling mechanism of subsystem modes. It also has the potential for forming SEA-like models of the system and predicting energy response and variance directly from coupled FE models of the individual subsystems, without the need for solution of the global eigenvalue problem.

INTRODUCTION

Statistical Energy Analysis (SEA), as an established subsystem- and energy-based approach, [1], is very appealing for high frequency vibration modelling. Compared to traditional deterministic Finite Element Analysis (FEA) [2], which is usually very detailed and expensive, SEA requires relatively very little computational cost. The paper presented here is aimed to predict the SEA-like response and variability of complex structures, and in particular systems comprising two subsystems coupled via an arbitrary interface, at higher frequencies in a more systematic way than conventional SEA. Free-interface component mode synthesis is used to develop the inter-modal coupling relations, which are seen to fall into three categories: (1) strongly connected, (2) weakly connected and (3) weakly coupled. In the case of weak connection/coupling, the inter-modal couplings reduce to two sets of stiffness-coupled oscillators (with modified damping values, if appropriate). The results can then be accommodated with a recently developed theory for coupled oscillators [3-6].

The paper is organized as follows. In section 2, the inter-modal dynamic coupling relations between the two sets of subsystem modes are quantified analytically. The concepts of strong and weak connection are discussed and the conditions of weak connection and weak coupling are distinguished. Effective mass and damping are introduced. In section 3 the newly developed coupled-oscillator approach is briefly reviewed. The results are directly applicable for weakly connected/coupled subsystems. In section 4, numerical examples are presented.

INTER-MODAL DYNAMIC COUPLING IN A BUILT-UP SYSTEM

In [7], a mode-based approach was developed to predict the vibration of two general, conservatively coupled subsystems as shown in Fig. 1 based on free-interface decomposition. The approach here is used to formulate the dynamic coupling relations between the modes of two coupled subsystems via an arbitrary interface.

The vibration equation of motion of subsystem a, in terms of a truncated set of free-interface component modes, can be written as

$$\left\{q_j^a\right\} = \left[Y_j^a\right]\left\{f_{e,j}^a + f_{I,j}^a\right\} \tag{1}$$

In the above equation, q_j^a is the jth modal amplitude of subsystem a, and $f_{e,j}^a$ and $f_{I,j}^a$ are respetively the modal forces corresponding the external excitation and the interface force acting on subsystem a, and $\left[Y_j^a\right]$ is the diagonal, modal receptance matrix, the jth diagonal element being given by

$$Y_j^a = \left[m_j^a\left(\left(\omega_j^a\right)^2\left(1+i\eta_j^a\right)-\omega^2\right)\right]^{-1} \tag{2}$$

Here m_j^a, ω_j^a and η_j^a are respectively the jth modal mass, natural frequency and loss factor. For simplicity, it is assumed $m_j^a = m_a$ and $\eta_j^a = \eta_a$. Similar expressions can be established for subsystem b by replacing the superscript a with b in Eqs. (1)-(2).

The interface force and displacement [6] are now decomposed in terms of a set of basis interface components ψ_p [8-9], with

$$\int_{V_I} \psi_p(V_I)\psi_{p'}(V_I)\,dV_I = \delta_{pp'} \tag{3}$$

Enforcing the displacement continuity and force equilibrium conditions along the interface [6], after some algebra, finally gives

$$\begin{Bmatrix} q_j^a \\ q_k^b \end{Bmatrix} = \begin{bmatrix} G_a & G_{ab} \\ G_{ab}^T & G_b \end{bmatrix} \begin{Bmatrix} f_{e,j}^a \\ f_{e,k}^b \end{Bmatrix} \tag{4}$$

where, the superscript T represents the transpose, and

$$[G_a] = \left[Y_j^a\right]\left[I - \alpha_{jp}\left(A_{pp} + B_{pp}\right)^{-1} \alpha_{jp}{}^T Y_j^a\right] \tag{5}$$

$$[G_b] = \left[Y_k^b\right]\left[I - \beta_{kp}\left(A_{pp} + B_{pp}\right)^{-1} \beta_{kp}{}^T Y_k^b\right] \tag{6}$$

$$[G_{ab}] = \left[Y_j^a\right]\left[\alpha_{jp}\left(A_{pp} + B_{pp}\right)^{-1} \beta_{jp}{}^T\right]\left[Y_k^b\right] \tag{7}$$

$$\left[A_{pp}\right] = \left[\alpha_{jp}\right]^T \left[Y_j^a\right]\left[\alpha_{jp}\right],\ \left[B_{pp}\right] = \left[\beta_{kp}\right]^T \left[Y_k^b\right]\left[\beta_{kp}\right] \tag{8}$$

$$\alpha_{jp} = \int_{V_I} \phi_j^a\left(V_I^a\right)\psi_p\left(V_I\right)dV_I\ ,\ \beta_{kp} = \int_{V_I} \phi_k^b\left(V_I^b\right)\psi_p\left(V_I\right)dV_I \tag{9}$$

ϕ_j^a and ϕ_k^b are the jth and kth mode shapes of subsystems a and b, respectively.

Eq. (4) indicates that $\left[G_a\right]$ is the Green function matrix which gives the modal amplitudes of subsystem a per unit modal forces applied to the same subsystem, and $\left[G_{ab}\right]$ the Green function matrix which gives the modal amplitudes of subsystem a per unit modal forces applied to subsystem b. The inter-modal dynamic stiffness coupling matrix can thus be determined analytically as

$$[D] = [G]^{-1} = \begin{bmatrix} D_a & D_{ab} \\ D_{ba} & D_b \end{bmatrix} \tag{10}$$

Eq. (10) helps to gain insight into the physics of the coupling mechanism of the subsystem modes, meanwhile, it allows the effects of uncertainty in the parameters of the subsystems and the interfaces to be included through Eqs. (5)-(8).

Coupling strength analysis: strong connection, weak connection, weak coupling

Physically, if some of the off-diagonal elements of $[D]$ are comparable to the diagonal elements, it indicates that the dynamic behaviour of each subsystem mode is significantly affected by both the mode set of the other subsystem and the rest of the modes within the same set of the subsystem. The two subsystems are then said to be *strongly connected*, in which case, full inter-modal coupling should strictly be considered if an accurate solution is required.

If $[D]$ is diagonally dominant, each subsystem mode remains largely uncoupled from the rest of the modes within the same set, in which case, the two subsystems can then be said to be *weakly connected*. This correspondingly gives, by the first-order perturbation of Eq. (10),

$$D_j^a \approx \left(Y_j^a\right)^{-1} + \left(\sum_p \sum_{p'} \alpha_{jp} \left(A+B\right)_{pp'}^{-1} \alpha_{jp'}\right) \tag{11}$$

$$D_{jk}^{ab} \approx \sum_p \sum_{p'} \alpha_{jp} \left(A+B\right)_{pp'}^{-1} \beta_{kp'} \tag{12}$$

Eq. (11) quantifies how the modal properties of the jth mode of subsystem a are modified by the coupling, and Eq. (12) formulates the coupling stiffness terms between the jth and the kth modes of subsystems a and b. The term $\left(A+B\right)_{pp'}^{-1}$ here accounts for the collective influence of the rest of the modes on the coupling of any particular pair of modes.

As suggested in [10], in the case of *weak coupling*, the modal properties of subsystems a and b are affected very little by each other after coupling. Then

$$\left[D_a\right] \approx \left[Y_j^a\right]^{-1},\ \left[D_b\right] \approx \left[Y_k^b\right]^{-1} \tag{13}$$

$$D_{jk}^{ab} \approx \sum_p \alpha_{jp} \left(A+B\right)_{pp}^{-1} \beta_{kp} \tag{14}$$

On the other hand, *weak coupling* is defined in [3-4] to occur if

$$\gamma^2 = \kappa^2 / \Delta^2 << 1,\ \kappa^2 = \mathrm{E}\left[\left|D_{jk}^{ab}\right|^2\right] \Big/ \left(m_a m_b \omega_c^2\right) \qquad (15), (16)$$

Here κ^2 is the connection strength parameter, and Δ is the damping bandwidth of the two subsystems within the frequency band of interest, and $\mathrm{E}[\bullet]$ the expectation, and ω_c the central frequency. Thus, the coupling strength decreases as the damping increases.

Effective mass and damping

For mass-normalised modes, Eq. (11) can be rewritten as

$$D_j^a \approx \left(\left(\omega_j^a\right)^2 \left(1 + i\eta_j^a\right) - \omega^2\right) + \mathrm{Re}\left\{D_j^{a'}\right\} + i\,\mathrm{Im}\left\{D_j^{a'}\right\} \tag{17}$$

The effective mass and effective damping induced to the jth mode of subsystem a, after coupling with subsystem b, can then be defined as

$$m_j^{a'} \approx \frac{1}{\omega^2} \mathrm{Re}\left\{-D_j^{a'}\right\},\ \eta_j^{a'} \approx \frac{1}{\omega^2} \mathrm{Im}\left\{D_j^{a'}\right\} \qquad (18), (19)$$

In Ref. [7], it was shown that both the effective mass and damping generally decrease as frequency increases. In the case of weak connection, $m_j^{a'} << m_a\left(=1\right)$, and for weak coupling, both $m_j^{a'} << m_a$ and $\eta_j^{a'} << \eta_a$.

THE COUPLED-OSCILLATOR THEORY

In [5-6], a theory was presented for the prediction of the ensemble mean and variance of the energy response of coupled oscillator sets, as shown in Fig. 2. The jth and kth oscillators of sets a and b are connected by a spring K_{jk}^{ab}. Assume that (1) the strength of connection between any pair of oscillators is sufficiently weak, (2) the properties of each set of oscillators are sufficiently random and (3) the modal overlap of each set is sufficiently high. Also for simplicity assume that each oscillator in set a has the same mass m_a, damping bandwidth Δ_a and ensemble average excitation spectral density S_f^a with similar assumptions for set b. Then under rain-on-the-roof excitation the ensemble mean energy of set a is

$$\bar{E}_a = \frac{\pi S_f^a}{m_a \Delta_a} n_a - \frac{\pi}{2} \frac{\kappa^2 n_a n_b}{\sqrt{\Delta^2 + \kappa^2}} \left(\frac{\pi S_f^a}{m_a \Delta_a} - \frac{\pi S_f^b}{m_b \Delta_b} \right) \quad (20)$$

where, $n_{a,b}$ are the modal densities of subsystems a and b, $\Delta = (\Delta_a + \Delta_b)/2$, and

$$\kappa^2 = \mathrm{E}\left[K_{jk}^{ab2} \right] \Big/ \left(m_a m_b \omega_c^2 \right) \quad (21)$$

is the parameter defining the strength of connection. The ensemble variance of the energy of set a, under the assumptions of Poisson and GOE natural frequency spacing statistics, can be predicted, respectively, as

$$Var\left[E_a\right] \approx \frac{n_a}{\Omega} \left(\frac{\pi S_f^a}{m_a \Delta_a} \right)^2 + \frac{\sigma_t}{\Omega} \left[\frac{\pi n_a n_b \kappa^4}{\left(2\sqrt{\Delta^2 + \kappa^2} \right)^3} \right] \left[\left(\frac{\pi S_f^a}{m_a \Delta_a} \right)^2 + \left(\frac{\pi S_f^b}{m_b \Delta_b} \right)^2 \right] \quad (22)$$

$$Var\left[E_a\right] \approx \frac{\ln\left(1 + \frac{\Omega^2}{\Delta_a^2}\right)}{\Omega^2 \pi^2} \left(\frac{\pi S_f^a}{m_a \Delta_a} \right)^2 + \frac{\sigma_t}{\Omega} \left[\frac{\pi n_a n_b \kappa^4}{\left(2\sqrt{\Delta^2 + \kappa^2} \right)^3} \right] \left[\left(\frac{\pi S_f^a}{m_a \Delta_a} \right)^2 + \left(\frac{\pi S_f^b}{m_b \Delta_b} \right)^2 \right] \quad (23)$$

In the above, Ω is the excitation bandwidth and

$$\sigma_t = \mathrm{E}\left[K_{jk}^{ab4} \right] \Big/ \mathrm{E}\left[K_{jk}^{ab2} \right]^2 \quad (24)$$

A rigorous calculation of σ_t remains an open issue. In [6], σ_t is estimated as

$$\begin{aligned} \sigma_t &\approx \left[\left(\mathrm{E}\left[\phi_a^4\right] \Big/ \mathrm{E}\left[\phi_a^2\right]^2 \right) \left(1 + \frac{1}{N_I} \right) \right] \left[\left(\mathrm{E}\left[\phi_b^4\right] \Big/ \mathrm{E}\left[\phi_b^2\right]^2 \right) \left(1 + \frac{1}{N_I} \right) \right]; \quad n\Delta < 1 \\ \sigma_t &\approx n\Delta \left[\left(\mathrm{E}\left[\phi_a^4\right] \Big/ \mathrm{E}\left[\phi_a^2\right]^2 \right) \left(1 + \frac{1}{N_I} \right) \right] \left[\left(\mathrm{E}\left[\phi_b^4\right] \Big/ \mathrm{E}\left[\phi_b^2\right]^2 \right) \left(1 + \frac{1}{N_I} \right) \right]; \; n\Delta \geq 1 \end{aligned} \quad (25)$$

where $n = (n_a + n_b)/2$, $\mathrm{E}\left[\phi_{a,b}^4\right] \Big/ \mathrm{E}\left[\phi_{a,b}^2\right]^2$ are the typical values of the statistical moments of the mode shapes of subsystems a and b (e.g. 3 for GOE mode shape statistics and 2.25 for sinusoidal modal shapes [11]), and N_I is the number of interface points. A fuller description is given in Ref. [6]. It is worth noting that, to account for the effective damping effects appropriately in the case of weak connection, Δ in Eqs. (20), (22)-(23) needs to be modified as $\Delta' = \left(\Delta'_a + \Delta'_b\right)/2$, $\Delta'_{a,b}$ being the in-situ effective modal bandwidths of the two subsystems.

NUMERICAL EXAMPLES

The foregoing analysis is implemented to predict the ensemble mean and variance of the energies of two coupled plates connected by a single point. Comparisons are made with those of Monte Carlo simulations of an ensemble of random structures. The system models considered comprise two plates. Plate 1 is rectangular with simply supported edges and an area of 0.32m^2. Plate 2 is a rectangular plate of an area 0.23m^2 but with one corner area of 0.03m^2 cut off. Each plate has a thickness of 3mm, with Young's modulus 10^{8}N/m^2, density 10^{3}kg/m^3, Poisson ratio 0.38 and damping loss factor 0.02. The frequency bandwidth employed is $\Omega = 0.2\omega$. Each plate is assumed to be with an uncertainty which can lead to a sufficiently large statistical overlap for the frequency range of interest. The Monte Carlo simulations are based on the solutions of a sufficient number of realizations of the system as an ensemble, the randomization of each plate being achieved by ‘‘geometry perturbation’’. An ensemble of 600 sample systems are generated which is sufficient enough for the convergence of the ensemble mean and a relative stable convergence of the ensemble variance [6]. The ensemble mean and variance of energy can then be calculated from the Monte Carlo simulations.

Fig. 3 compares the ensemble mean of energy of each plate. It is seen that the present theory tends to agree well with the Monte Carlo solution for $\omega > 750 rad/s$. (Above this frequency, the induced effective damp loss factors for both plates are less than a third of the nominal damping loss factor, and thus the two plates can be treated as weakly coupled.) The apparent fluctuations of the Monte Carlo solutions at lower frequencies are due to lack statistical overlap of the ensemble examples. They gradually improved at higher frequencies as the system modal overlap increases. The poor agreements at very low frequencies are due to the strong coupling effects between the two plates in which case the main assumption to maintain the validity of the predicting theory has broken down.

Fig. 4 compares the relative variance of energy of each plate. Comparing to the Monte Carlo solutions, Eq. (22) behaves significantly better than Eq.(23) for plate 1 (conforming to Poisson statistics [6]); and for plate 2 (conforming to GOE statistics [6]) the latter equation has a better performance than the former one. This may suggest that, in the weakly coupling region, the ensemble variance of energy of each subsystem is dominated by only the statistics of the subsystem itself but relatively less sensitive to those of the rest of the system. Similar to Fig. 3, Fig. 4 shows that the present theory is not in good agreement with the Monte Carlo solution at low frequencies. This may well explained as partly due to the strong coupling limitation of the present theory, and partly due to the inevitably lack of statistical overlap.

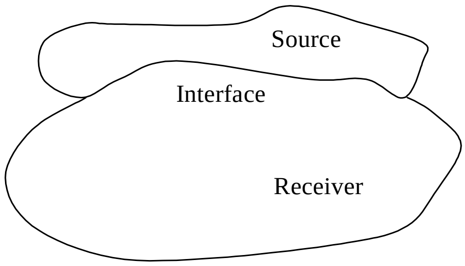

Figure 1 – Two subsystems connected via an arbitrary interface

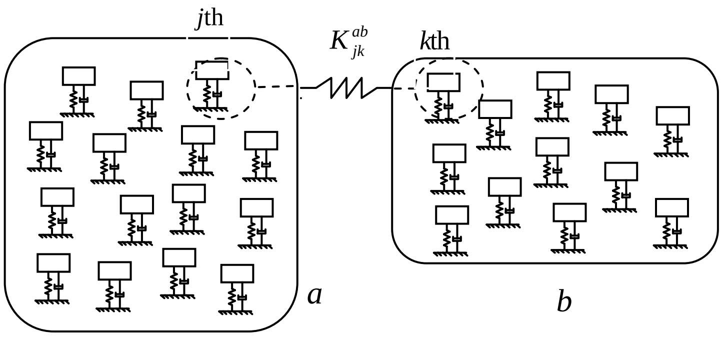

Figure 2 – Two spring-coupled sets of oscillators

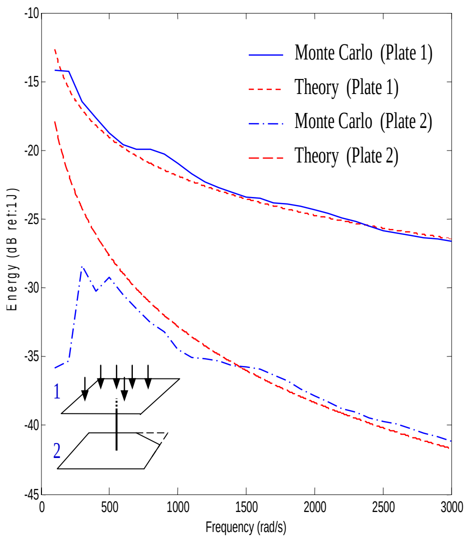

Figure 3 – Ensemble mean energy

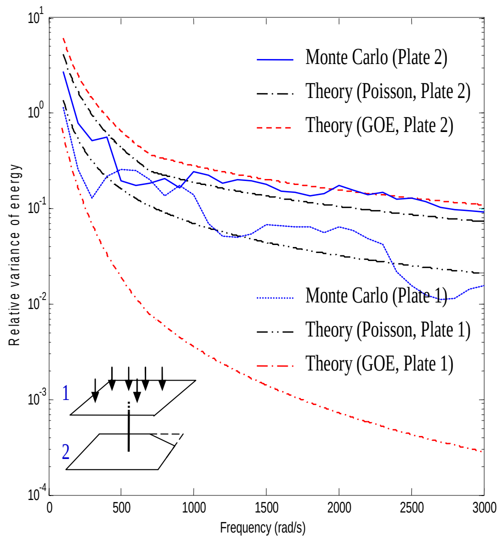

Figure 4 – Relative variance of energy

SUMMARY

In this paper, the inter-modal coupling relations are established for built-up structures comprising two subsystems connected via an arbitrary interface based on free-interface component mode synthesis. The dynamic coupling between the two subsystems is seen to fall into 3 categories as (1) strongly connected, (2) weakly connected and (3) weakly coupled. Effective mass and damping are introduced to account for the dynamic influence on a particular subsystem mode of the remaining modes after the subsystems are coupled. In the sense of weak connection/coupling, the inter-modal coupling relations reduce to those of two sets of stiffness-coupled oscillators (with modified damping values, if appropriate). As a result, a coupled oscillator theory can be used to predict the ensemble mean and variance of the energy of each subsystem. Numerical examples are given for a system of two plates connected at a single point. The approach is seen to agree well when the two subsystems are weakly coupled.

The approach helps to give insight into the physics of the coupling mechanism of subsystem modes. It also has the potential for forming SEA-like models of the system and predicting ensemble mean energy and variance directly from coupled FE models of the individual subsystems, without the need for solution of the global eigenvalue problem.

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