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CHAOTIC MOTION OF A HEATED BIMETALLIC THIN CIRCULAR PLATE

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Abstract

Considering the effect of geometric nonlinearity and uniformly distributed stationary temperature, the bifurcation behaviors and chaotic phenomena of a bimetallic thin circular plate under transverse periodic excitation are investigated in this paper. First of all, the nonlinear dynamic equations for the bimetallic plate are established by employing the Galerkin's technique. Furthermore, the critical conditions for occurrence of homoclinic and subharmonic bifurcations as well as chaos are studied theoretically by means of Melnikov function method. Finally, the chaotic motions are searched and simulated numerically with the application of Computer Algebra Systems Maple, and the Poincaré map and phase curve along with time-history diagram are used to evaluate if a chaotic motion appears. The results indicate that there exist some chaotic motions in heated bimetallic plate.

INTRODUCTION

The present discussed bimetallic plates are widely used in precision instruments and micromachines. Much attention has received for the thermal stability problem of this kind of plates and shells[4], [5]. However, there are few archival publications related to their chaotic motion and bifurcation behavior to the best of authors' knowledge. By the redetermination of reference surface of coordinate, the authors obtained the compact control equations recently for the nonlinear vibration problem of heated thin bimetallic plates and further gained their periodic solutions, but still with no concern of their chaotic motion[6], [7].

In fact, the nonlinear dynamic system related to elastic structure has received great attention to the forced vibrations and model analyses for beams, however, the chaotic motions and bifurcation behaviors of plates and shells are not well established

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to the best of authors' knowledge, relatively little work has been done on them[1], [8]. Based on the governing equations set up by authors in former works[6], [7], the chaotic motion of a thin circular bimetallic plate under combined uniformly distributed stationary temperature and transverse periodic excitation are investigated. The critical conditions of bifurcations and chaos are given theoretically and the chaotic motions are simulated numerically.

DYNAMIC BASIC EQUATIONS

Consider a bimetallic circular plate with its total thickness h small in comparison with radius a is composed of two thin homogeneous isotropic metallic plates bonded at the common surface, such that no slippage can occur. The clamped immovable edge condition for plate under uniformly distributed stationary temperature T and transverse excitation $q\cos\omega t$ is considered and the material properties of the plate are assumed to be independent of temperature.

Let h_i, ρ_i, E_i and α_i be the thickness, mass density, Young's modulus and thermal expansion coefficient of each layer. Here, $i = 1,2$ represent the upper and lower layer respectively. Assuming Poisson's ratio $\nu_i = \nu$ [4], then based on Von Kármán's theory, the dimensionless equations governing the axisymmetrically large amplitude vibration of a circular bimetallic plate can be derived from Hamilton's principle as follows

$$\nabla^4 W + \frac{\partial^2 W}{\partial \bar{\tau}^2} + \mu \frac{\partial W}{\partial \bar{\tau}} = L(W\ ,\Phi) + Q\cos\varpi\bar{\tau} \tag{1}$$

$$\nabla^4 \Phi = -\frac{1}{2} L(W\ ,W) \tag{2}$$

$$\text{at}\quad R = 0,\ W < \infty,\ \frac{\partial W}{\partial R} = 0,\ \frac{1}{R}\frac{\partial \Phi}{\partial R} < \infty \tag{3}$$

$$\text{at}\quad R = 1,\ W = 0,\ \frac{\partial W}{\partial R} = 0,\ \frac{\partial^2 \Phi}{\partial R^2} - \frac{\nu}{R}\frac{\partial \Phi}{\partial R} + \lambda = 0 \tag{4}$$

These governing equations for double-layered plates are similar to those of classical single-layered plates theory. The dimensionless quantities are related to the corresponding physical ones through the following relations

$$R = r/a\ ,\ W = w\sqrt{C\left(1-\nu^2\right)/D}\ ,\Phi = \varphi/D\ ,\lambda = (1-\nu)\alpha_m a^2 T/D\ ,\bar{\tau} = t\sqrt{D/\rho h a^4}$$

$$\mu = \delta a^2 \big/ \sqrt{\rho h D}\ ,\quad \varpi = \omega\sqrt{\rho h a^4/D}\ ,\quad Q = q a^4 \sqrt{C\left(1-\nu^2\right)/D^3}$$

in which, r is the radial coordinate, t the time variable, δ the damping coefficient, w the deflection of reference surface, φ the stress function, ∇^4 and L are two partial differential operators with respect to R, and C, D, α_m, ρ signify the effective extensional rigidity, flexural rigidity, first order thermal expansion coefficient and mass density, respectively(see references [6], [7] for detailed definitions).

The following single mode expression for W, in the usual way, is assumed

$$W(R,\bar{\tau}) = A(\bar{\tau})\left(1-R^2\right)^2 \tag{5}$$

which has already satisfied the boundary conditions of W in equations (3-4). Substituting the equations (5) into compatibility equation (2) and noting the boundary condition of φ, the solution for stress function may be arrived as

$$\frac{\partial\Phi}{\partial R} = -\frac{\lambda}{1-\nu}R + \left[\frac{5-3\nu}{6(1-\nu)}R - R^3 + \frac{2}{3}R^5 - \frac{1}{6}R^7\right]A^2(\bar{\tau}) \tag{6}$$

Substitution of equations (5-6) into equation (1), and application of Galerkin's method yield a nonlinear differential equation for A as

$$\frac{\mathrm{d}^2 A}{\mathrm{d}\bar{\tau}^2} + \mu\frac{\mathrm{d}A}{\mathrm{d}\bar{\tau}} - \alpha A + \gamma A^3 = \frac{5}{3}Q\cos\varpi\bar{\tau} \tag{7}$$

This is a dynamic system of Duffing type with

$$\alpha = \frac{20(\lambda-\lambda_{cr})}{3(1-\nu)}, \gamma = \frac{10(23-9\nu)}{63(1-\nu)}$$

Here, $\lambda_{cr} = 16(1-\nu)$ is the critical dimensionless temperature at which the plate is in buckled state.

ANALYSIS OF THE CHAOTIC MOTION

The following discussions on chaotic problems of nonlinear dynamic equation are carried out only for the case of $\alpha > 0$. To this end, introducing the new transformations

$$A = x\sqrt{\frac{\alpha}{\gamma}}, \bar{\tau} = \frac{\tau}{\sqrt{\alpha}}, \ \varpi = \Omega\sqrt{\alpha}, \mu = \varepsilon\eta\sqrt{\alpha}, \ Q = \frac{3}{5}\varepsilon f\alpha\sqrt{\frac{\alpha}{\gamma}}$$

and denoting $\dot{x} = y$, here dot defines the differentiation with respect to τ, then equation (7) can be rewritten to the following

$$\begin{cases} \dot{x} = y \\ \dot{y} = x - x^3 + \varepsilon\left(f\cos\Omega\tau - \eta y\right) \end{cases} \tag{8}$$

Qualitative analysis for unperturbed system

Apart from the ε -term, equations (8) becomes a unperturbed Hamilton system, with Hamiltonian

$$H(x,y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4 = h \tag{9}$$

Obviously, this system has one hyperbolic saddle point at $(0,0)$ and two centers at $(\pm1,0)$ in phase portrait. For different values of Hamiltonian, the system indicates different dynamic behavior. For $h=0$, one obtains the two homoclinic orbits (here and henceforth, only the orbits in right half of phase space are discussed)

$$\begin{cases} x_o(\tau) = \sqrt{2}\operatorname{sech}\tau \\ y_o(\tau) = -\sqrt{2}\tanh\tau\operatorname{sech}\tau \end{cases} \tag{10}$$

When $k \in (0,1)$ satisfies $h(k) = (k^2-1)/(2-k^2)^2$, one gets a one-parameter family of periodic orbits within each of homoclinic orbit

$$\begin{cases} x_k(\tau) = \dfrac{\sqrt{2}}{\sqrt{2-k^2}}\operatorname{dn}\left(\dfrac{\tau}{\sqrt{2-k^2}},k\right) \\ y_k(\tau) = -\dfrac{\sqrt{2}k^2}{2-k^2}\operatorname{sn}\left(\dfrac{\tau}{\sqrt{2-k^2}},k\right)\operatorname{cn}\left(\dfrac{\tau}{\sqrt{2-k^2}},k\right) \end{cases} \tag{11}$$

the period of these orbits is $T_k = 2\sqrt{2-k^2}K(k)$. Here sn, cn and dn are the Jacobi elliptic functions and k is the elliptic modulus, $K(k)$ is the complete elliptic integral of the first kind. It may be verified that $\mathrm{d}T_k/\mathrm{d}k > 0$, namely, T_k increases monotonically with k, when $k \to 1$, $K(k) \to \infty$, T_k will approach infinity as a limit.

if $k \in \left(1/\sqrt{2},1\right)$ satisfies $h(k) = k^2(1-k^2)/(2k^2-1)^2$, one gets another one-parameter family of periodic orbits outside the homoclinic orbit[2]

$$\begin{cases} x_k(\tau) = \dfrac{\sqrt{2}k}{\sqrt{2k^2-1}}\operatorname{cn}\left(\dfrac{\tau}{\sqrt{2k^2-1}},k\right) \\ y_k(\tau) = -\dfrac{\sqrt{2}k}{2k^2-1}\operatorname{sn}\left(\dfrac{\tau}{\sqrt{2k^2-1}},k\right)\operatorname{dn}\left(\dfrac{\tau}{\sqrt{2k^2-1}},k\right) \end{cases} \tag{12}$$

now the orbits period becomes $T_k = 4\sqrt{2k^2-1}K(k)$, and still $\mathrm{d}T_k/\mathrm{d}k > 0$.

Melnikov function and bifurcation for perturbed system

For a pair of given prime integers m and n, the Melnikov function of subharmonic orbits satisfies the resonance condition $T_k = 2m\pi/n\Omega$ in perturbed system (8) is expressed by[3]

$$M^{\frac{m}{n}}(\tau_0) = \int_0^{nT_k} \left[-\eta y_k(\tau) + f\cos\Omega(\tau+\tau_0)\right] y_k(\tau)\mathrm{d}\tau = -\eta J_1(m,n) + fJ_2(m,n)\sin\Omega\tau_0 \quad (13)$$

where τ_0 is the reference time. When the parameters η and f satisfies

$$\frac{f}{\eta} > \frac{J_1(m,n)}{J_2(m,n)} = R_m(\Omega) \quad (14)$$

the subharmonic periodic solution exists and subharmonic bifurcation occurs in system. Here $R_m(\Omega)$ defines the threshold value for subharmonic periodic solution of order m. For the Melnikov function of periodic orbits (11), one has

$$J_1(m,n) = \frac{4}{3\sqrt{2-k^2}}\left[E(k) - 2\frac{1-k^2}{2-k^2}K(k)\right] \quad (15)$$

$$J_2(m,n) = \begin{cases} 0 & , \ n \neq 1 \\ \sqrt{2}\pi\Omega\,\mathrm{sech}\left[\Omega\sqrt{2-k^2}K'(k)\right] & , \ n = 1 \end{cases} \quad (16)$$

Here $K'(k) = K(k') = K\left(\sqrt{1-k^2}\right)$, $E(k)$ is the complete elliptic integral of the second kind. From the above two equations, one can concludes that when $n=1$ and $f/\eta > J_1(m,1)/J_2(m,1) = R_m^{(1)}(\Omega)$, the subharmonic periodic solution of order m exists in the system. With an analogous analysis for periodic orbits (12), in the present case

$$J_1(m,n) = \frac{8}{3\sqrt{2k^2-1}}\left[E(k) + \frac{1-k^2}{2k^2-1}K(k)\right] \quad (17)$$

$$J_2(m,n) = \begin{cases} 0 & , \ n \neq 1 \text{ or } m \textit{ is even} \\ 2\sqrt{2}\pi\Omega\,\mathrm{sech}\left[\Omega\sqrt{2k^2-1}K'(k)\right] & , \ n = 1 \text{ and } m \textit{ is odd} \end{cases} \quad (18)$$

Thus when $n=1$ and $f/\eta > J_1(m,1)/J_2(m,1) = R_m^{(2)}(\Omega)$, the subharmonic periodic solution with odd order exists in the system. In a similar manner, the Melnikov function for homoclinic orbits (10) is easily given and explicitly computed by

$$M\left(\tau_0\right)=\int_{-\infty}^{+\infty}\left[f\cos\Omega\left(\tau+\tau_0\right)-\eta y_0\left(\tau\right)\right]y_0\left(\tau\right)\mathrm{d}\tau=-\frac{4}{3}\eta+\sqrt{2}\pi\Omega f\operatorname{sech}\frac{\pi\Omega}{2}\sin\Omega\tau_0 \tag{19}$$

So if and only if

$$\frac{f}{\eta}\geq\frac{2\sqrt{2}}{3\pi\Omega}\cosh\frac{\pi\Omega}{2}=R^0\left(\Omega\right) \tag{20}$$

the homoclinic orbits exist for which Melnikov function has a simple zero and consequently chaotic motions in the sense of Smale horseshoe may occur. Here $R^0\left(\Omega\right)$ is defined as the threshold value for horseshoe transformation.

From the above theoretical analysis one see, only subharmonic bifurcation of $n=1$ will appear. Noting that in orbits (11) and (12), when $m\to\infty$ (namely $k\to 1$), $h\to 0$, the following relation exists for each fixed Ω

$$\lim_{m\to\infty}R_m^{(1)}\left(\Omega\right)=\lim_{m\to\infty}R_m^{(2)}\left(\Omega\right)=R^0\left(\Omega\right) \tag{21}$$

which means that the thresholds of subharmonic bifurcation will tend to the thresholds of horseshoe transformation for a periodically excited bimetallic plate with f/η increasing gradually, subharmonic bifurcation reach its limit when Smale horseshoe occurs.

NUMERICAL RESULTS AND DISCUSSIONS

It is instructive to examine the behavior of the chaotic threshold R^0 as functions of the temperatures parameters λ and excitation frequencies parameters Ω or ϖ. A typical plot of R^0 vs. ϖ for some fixed values of λ is shown in Fig. 1, from which one sees that $R^0\left(\varpi\right)$ graph exhibits a similar shape of parabola. Furthermore, R^0 has a single minimum at $\varpi_{\min}$, the most chaotic frequency. This value can be computed exactly by solving the transcendental equation $\mathrm{d}R^0/\mathrm{d}\Omega=0$ for Ω, which gives the value of $\Omega=0.763739$ or $\varpi=0.763739\sqrt{\alpha}$ as the root of this transcendental equation, and further the minimum of R^0 be found as $R^0\left(\Omega\right)=0.711293$. Fig. 1 also permits understanding how the parameter λ influences the chaotic parameter region, for low values of excitation frequency, λ does not affect R^0 appreciably; for high values of excitation frequency, the probability of chaotic motion is increase with λ.

Further investigations for equation (8) are developed by means of computer simulation with the application of Computer Algebra Systems Maple to find the possible chaotic solution. The Poincaré map and phase portrait as well as time-displacement history technique are examined and the chaotic response is distinguished in this way from a regular one. A special group of dimensionless parameters include $\nu=0.3$, $\lambda=1.2\lambda_{cr}$, $\varpi=2.31$, $Q=25$, $\mu=0.5$ are taken as an example, and the criterion of Melnikov is satisfied in this case. The corresponding system features are numerically simulated with 6000 computation points and depicted in Fig. 2. It is found from Fig. 2 that the chaotic characteristic appears, the time-displacement history shown in Fig. 2(a) is irregular, the phase portrait in Fig. 2(b) is intertwisted, neither repeatable nor regular, the Poincaré map in Fig. 2(c) reflects a complex chaotic attractor, thus we say that this is chaotic motion.

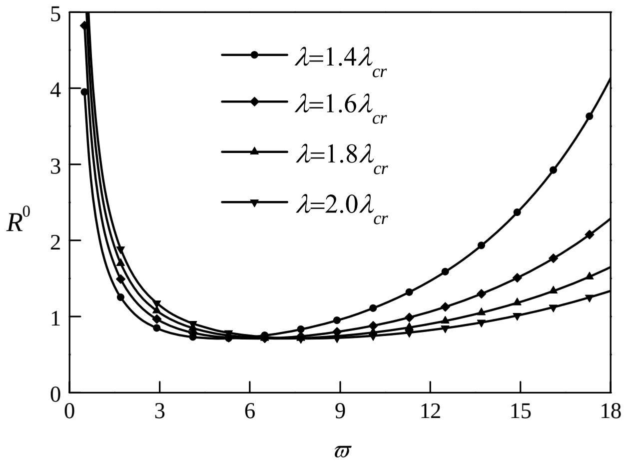

Figure 1 – $R^0 \sim \varpi$ graph

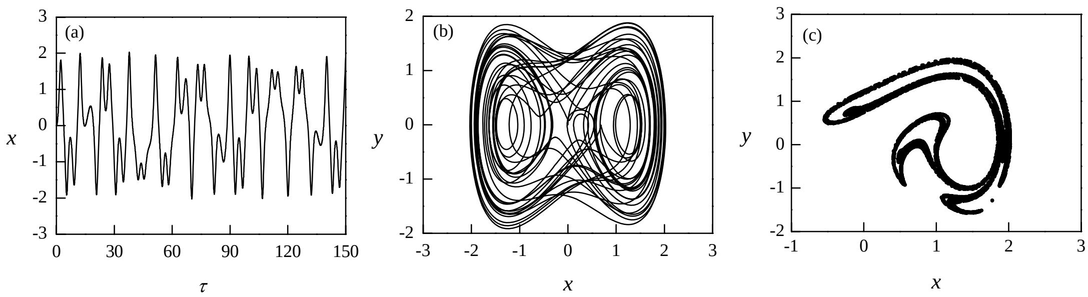

Figure 2 –Chaotic motion: (a) time –displacement history (b) phase portrait (c) Poincaré map

The results of numerical simulation illustrate that a large deflection motion of the heated bimetallic plate possess complex aperiodic behavior under various values of μ, λ, Q, ϖ. for example, a period-2 motion will be obtained when other parameters remain above and only changes μ to 2.85, see Fig.3 for this result.

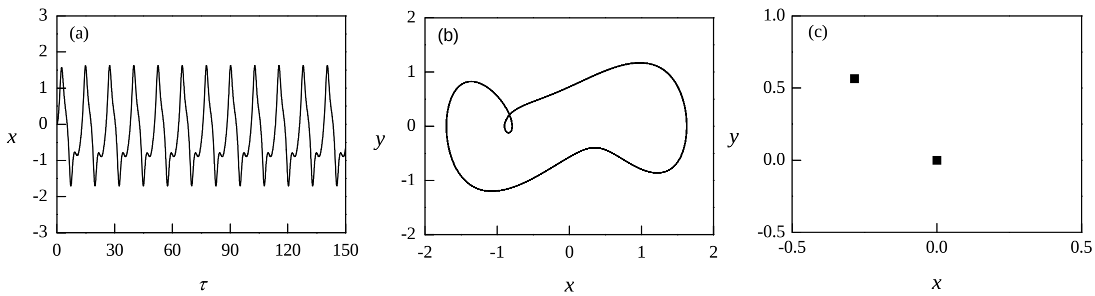

Figure 3 –Period motion: (a) time –displacement history (b) phase portrait (c) Poincaré map

SUMMARY

The chaotic motion of a bimetallic circular plate with transverse periodic excitation was investigated considering the effect of large deflection and uniformly stationary temperature. The theoretical critical conditions for bifurcations and chaos were determined and the complex aperiodic behaviors were simulated. The results identify conditions for which chaotic motion occurs in bimetallic plates.

The present work can easily be degenerated into the chaotic analysis for heated thin single plates or expanded to the similar study for heated multi-layered plates.

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