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DISPERSION OF ULTRASONIC WAVES IN TRANSVERSELY ISOTROPIC CYLINDERS

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Abstract

The problem of propagation of flexural guided waves in infinite, homogeneous, transversely isotropic circular cylinders is studied within the framework of the linear three-dimensional theory of elasticity. For solving the equations of motion, the displacement field is expressed in terms of three perpendicular scalar potential functions each representing one of the compression (P), vertically-polarized shear (SV), and horizontally-polarized shear (SH) waves. This representation results in complete decoupling of the equations of motion for the SH wave while P and SV wave equations remain coupled. This is in agreement with the physics of the aforementioned wave modes. The frequency equation for the propagation of flexural guided waves in free transversely isotropic cylinders is developed and numerically solved. The model is used in solving the frequency equation of isotropic cylinders and several transversely isotropic cylinders. The results obtained for isotropic materials are used as benchmarks and completely agree with those obtained from other mathematical models. Since the results obtained for transversely isotropic materials are the first of their kind, there is no means of comparison. The validity of these frequency spectra is investigated by comparing them with their counterparts calculated for isotropic cylinders.

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INTRODUCTION

The description of acoustic propagation in a composite media is of great interest in a variety of applications such as nondestructive evaluation (NDE) and acoustic design, and there is need for a flexible model that can handle the propagation of acoustic waves in these media. In this paper a systematic approach to this question will be presented.
The problem of acoustic waves propagating in isotropic cylinders has been treated by Pochhammer [1] in 1876, who derived the dispersion equation for waves propagating in circular cylinders. Various researchers [2,3] have used the dispersion equations derived by Pochhammer to develop the cases of short wavelength and long wavelength limits. Hudson [3] has developed the theory of elastic vibrations in solid cylinders of homogeneous isotropic materials. He has studied both longitudinal and flexural waves. Pao and Mindlin [4] and Pao [5] have also studied the dispersion of flexural waves in isotropic cylinders.
Morse used the series method [6] to extend the Pochhammer's solution to the case of transversely isotropic cylinders. Mirsky used the potential functions method [7] for the same purpose. Subsequently, several other researchers examined the propagation of longitudinal waves in transversely isotropic cylinders, for example Xu and Datta [8], Dayal [9] and Nagy [10]. Ahmad [11] have also examined the propagation of flexural waves in cylinders immersed in a fluid using the method proposed by Mirsky.
It can be stated that for the problem of wave propagation in isotropic cylinders, a systematic solution has been established; however, this is not true for wave propagation in transversely isotropic cylinders. In all previous models for anisotropic cylinders, the form of the solution is somehow guessed a priori. In the work reported here, a new approach is proposed in which the flexural modes of an infinitely long free transversely isotropic cylinder are found through a systematic approach. Using this model, the frequency equations are solved and the dispersion curves are plotted for a number of isotropic and transversely isotropic cylinders.

THEORY

Consider a transversely isotropic circular cylinder with its axis coincident with z-axis of a cylindrical coordinate system.

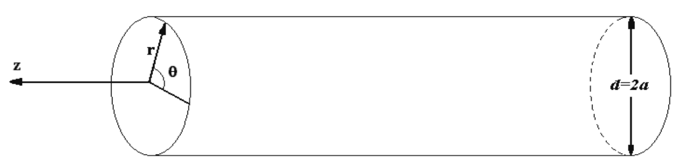

Figure 1 - The coordinate system used in the derivation of equations.

The constitutive relation for such material, when the transverse isotropy is in the $r-\theta$ plane, can be written as,

$$\begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \sigma_{z\theta} \\ \sigma_{rz} \\ \sigma_{r\theta} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & (c_{11}-c_{12})/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{zz} \\ \gamma_{rz} \\ \gamma_{\theta z} \\ \gamma_{r\theta} \end{Bmatrix} \tag{1}$$

where $\sigma_{ij}\,(i, j = r, \theta, z)$ is the stress tensor and ε_{ij} and γ_{ij} are, respectively, normal and engineering shear components of the strain tensor; c_{ij} denote the five independent elastic constants of a transversely isotropic material.

The equation of motion, in the absence of body forces, is given by

$$\sum_{j=1}^{3} \sigma_{ij,j} = \rho \ddot{U}_i . \tag{2}$$

where U and ρ are the displacement vector and density of the cylinder, respectively. Substituting Eq. (1) in Eq. (2), gives a set of three wave equations. These equations are in terms of u_r, u_θ and u_z which are the components of the displacement vector.

According to the second theorem of Helmholtz, the displacement field can be decomposed into a longitudinal irrotational field and a transverse solenoidal field which satisfy the governing equations. Therefore, the displacement vector is written in terms of three scalar potentials functions as follows [12],

$$U = \nabla\phi + \nabla\times(\chi\,\hat{e}_z) + a\nabla\times\nabla\times(\psi\,\hat{e}_z). \tag{3}$$

where ϕ represents the P wave (compression wave), ψ and χ, respectively, represent the SV wave (vertically polarized shear wave) and SH wave (horizontally polarized shear wave), $\nabla\phi$ is the rotation-free field (curl of the gradient is zero) and $\nabla\times(\chi\,\hat{e}_z) + a\nabla\times\nabla\times(\psi\,\hat{e}_z)$ is the solenoidal field (divergence of the curl is zero). By substituting Eq. (3) in the wave equations, these wave equations are expressed in terms of potential functions [13],

$$\left(\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\Bigg\{ c_{11}\nabla^2\phi + (c_{13} + 2c_{44} - c_{11})\frac{\partial^2\phi}{\partial z^2} - \rho_c \frac{\partial^2\phi}{\partial t^2} + \\ a\frac{\partial}{\partial z}\left[(c_{11} - c_{13} - c_{44})\nabla^2\psi + (c_{13} + 2c_{44} - c_{11})\frac{\partial^2\psi}{\partial z^2} - \rho_c \frac{\partial^2\psi}{\partial t^2} \right]\Bigg\} = 0 \tag{4}$$

$$\frac{\partial}{\partial z}\left[(c_{13}+2c_{44})\nabla^2\phi+(c_{33}-c_{13}-2c_{44})\frac{\partial^2\phi}{\partial z^2}-\rho_c\frac{\partial^2\phi}{\partial t^2}\right]+$$
$$a\left(\frac{\partial^2}{\partial z^2}-\nabla^2\right)\left[c_{44}\nabla^2\psi+(c_{33}-c_{13}-2c_{44})\frac{\partial^2\psi}{\partial z^2}-\rho_c\frac{\partial^2\psi}{\partial t^2}\right]=0 \quad (5)$$

$$\left(\nabla^2-\frac{\partial^2}{\partial z^2}\right)\left[\frac{(c_{11}-c_{12})}{2}\nabla^2\chi+\left(c_{44}-\frac{(c_{11}-c_{12})}{2}\right)\frac{\partial^2\chi}{\partial z^2}-\rho_c\frac{\partial^2\chi}{\partial t^2}\right]=0 \quad (6)$$

The mathematical problem described by Eqs. (4)-(6) and appropriate boundary conditions on the lateral surfaces provide a well-posed mathematical problem that can be solved by the method of separation of variables in a systematic manner. We will solve one of these equations here.

Applying the separation of variables technique, the solution to Eq. (6) can be obtained by setting the terms in square brackets equal to zero, i.e.,

$$\frac{(c_{11}-c_{12})}{2}\nabla^2\chi+\left(c_{44}-\frac{(c_{11}-c_{12})}{2}\right)\frac{\partial^2\chi}{\partial z^2}-\rho_c\frac{\partial^2\chi}{\partial t^2}=0. \quad (7)$$

By solving the resulting separated partial differential equations, χ is found to be of the form [14],

$$\chi(r,\theta,z,t)=\sum_{n=0}^{\infty}D_nJ_n(s_3\,r)\sin(n\theta)e^{i(k\,z-\omega t)}. \quad (8)$$

Following a similar procedure and taking advantage of the irrotational and solenoidal properties of the longitudinal and transverse fields, it can be shown that,

$$\phi(r,\theta,z,t)=\sum_{n=0}^{\infty}(B_nJ_n(s_1\,r)+q_2C_nJ_n(s_2 r))\cos(n\theta)e^{i(k\,z-\omega t)}, \quad (9)$$

$$\psi(r,\theta,z,t)=\sum_{n=0}^{\infty}(q_1B_nJ_n(s_1 r)+C_nJ_n(s_2\,r))\cos(n\theta)e^{i(k\,z-\omega t)}, \quad (10)$$

where $k=\omega/c$ is the wave number, c is the phase velocity, ω is the circular frequency and J_n are first type Bessel functions of order *n*. Moreover, s_1, s_2, s_3, q_1 and q_2 are constants which depend on the elastic constants of the material as well as the frequency and wave number [13].

For a cylinder in vacuum, the traction-free boundary conditions hold. Therefore, at *r=a*:

$$\sigma_{rr} = \sigma_{r\theta} = \sigma_{rz} = 0 \tag{11}$$

Expanded expressions for the stress and displacement at any point can be derived in terms of potential functions. Inserting the potential functions of Eqs. (8)-(10) into Eq. (11) results in the following system of linear algebraic equations:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} A_n \\ B_n \\ C_n \end{bmatrix} = 0 \tag{12}$$

The expressions for a_{ij} are given in the Appendix. The solution to Eq. (12) is nontrivial only if the determinant of the coefficients vanishes. Therefore,

$$\det(a_{ij}) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0 \tag{13}$$

Equation (13) represents the frequency equation for flexural waves propagating along a transversely isotropic cylinder in vacuum. Using this equation, the dispersion curves can be plotted for different modes.

NUMERICAL RESULTS

The wave numbers of the flexural guided modes can be calculated at any frequency by numerically searching for the zeros of Eq. (13). We shall present the frequency spectrum in terms of the frequency-dependent wave number. The frequency spectrum can be plotted using the roots of Eq. (13). Elastic properties of aluminum, magnesium, and sapphire are given in table 1.

Table 1, Physical parameters.

Material	Stiffness $\times 10^{11}$ (N/m^2)					Density (kg/m^3)
	c_{11}	c_{12}	c_{13}	c_{33}	c_{44}	
aluminum	1.108	0.612	0.612	1.108	0.249	2690
magnesium	0.597	0.262	0.217	0.617	0.164	1740
sapphire	4.968	1.636	1.109	4.981	1.474	3986

Figure 2 shows the calculated frequency curves for different flexural modes of aluminum. These curves are identical to those presented by Pao in Figure 4 of Ref. 5. The corresponding curves for magnesium and sapphire with transversely isotropic elastic properties are shown in Figs. 3 and 4, respectively. The general behavior of the frequency curves of transversely isotropic materials is similar to those of aluminum;

however, no similar results exist in the literature for comparison. The vertical frequency axes in Figs. 2-4 are normalized with respect to c_b where $c_b = \sqrt{E_a / \rho}$ and E_a is the axial Young's modulus defined as $E_a = c_{33} - 2c_{13}^2 / (c_{11} + c_{12})$. The intersection of a mode curve contour with the frequency axis (*ka*) indicates a cut-off in the sense that it is a propagation limit: i.e., a resonance with infinite wavelength. In the case of the lowest mode for aluminum, the low-frequency limit for the phase velocity is zero and also the high-frequency limit of the phase velocity is the Rayleigh wave velocity.

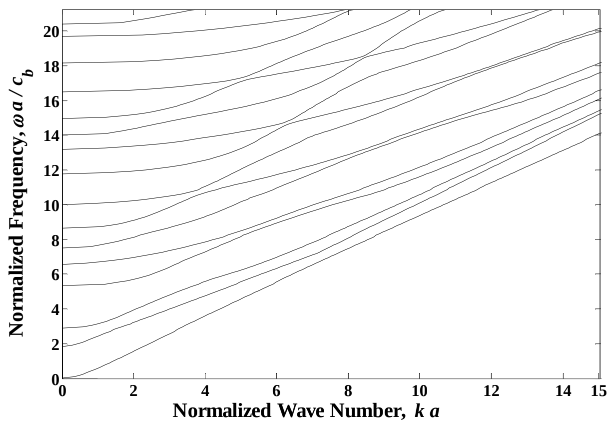

Figure 2 - Frequency curves for the flexural modes of a homogeneous isotropic aluminum cylinder.

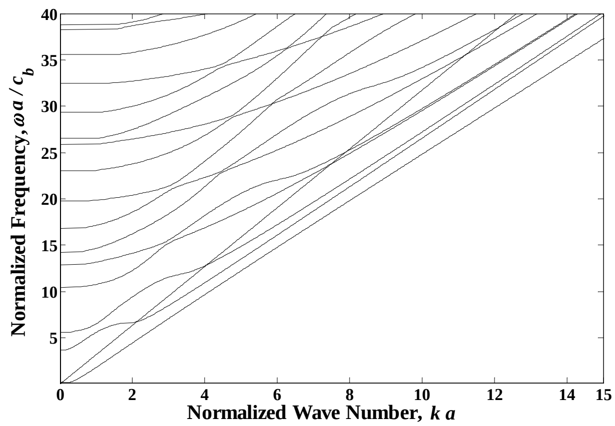

Figure 3 - Frequency spectrum for the flexural modes of a homogeneous transversely isotropic magnesium cylinder.

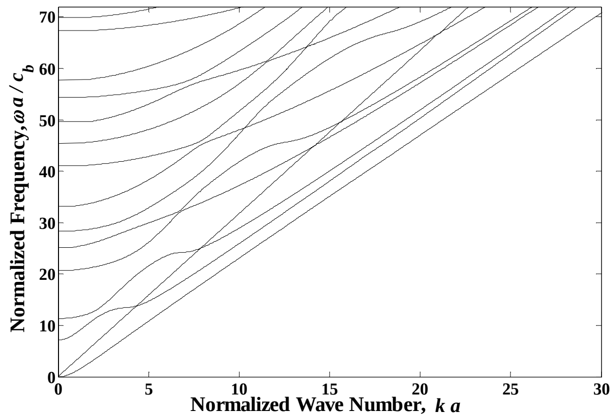

Figure 4 - Frequency curves for the flexural modes of a homogeneous transversely isotropic sapphire cylinder.

CONCLUSIONS

In this paper, the propagation of flexural guided waves in infinite, homogeneous, transversely isotropic circular cylinders was studied. The problem was solved by introducing a displacement field representation for solving the governing wave equations, which decouple the equations for the quasi P-SV and SH waves, and makes the separation of variables possible in cylindrical coordinates. In the numerical example, by applying appropriate boundary conditions, the frequency spectrum of isotropic aluminum was plotted as a benchmark. Frequency spectra of a number of transversely isotropic materials were also calculated and plotted. This method of solution can also be extended to solving the problems of wave propagation in free, immersed and encased shells and multilayered cylinders.

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APPENDIX

Elements of the matrices given in Eq. (6),

$$a_{11} = [c_{11} + ikaq_1(c_{11} - c_{13})][(n^2 - n - s_1{}^2 a^2)J_n(s_1 a) + (s_1 a)J_{n+1}(s_1 a)]$$
$$+[c_{12} + ikaq_1(c_{12} - c_{13})][n^2 J_n(s_1 a) - (s_1 a)J_{n+1}(s_1 a)] + \left(-c_{13}k^2 a^2 - c_{12}n^2 + in^2 kaq_1(c_{13} - c_{12})\right)J_n(s_1 a),$$
$$a_{12} = [c_{11}q_2 + ika(c_{11} - c_{13})][(n^2 - n - s_2{}^2 a^2)J_n(s_2 a) + (s_2 a)J_{n+1}(s_2 a)]$$
$$+[c_{12}q_2 + ika(c_{12} - c_{13})][n^2 J_n(s_2 a) - (s_2 a)J_{n+1}(s_2 a) + \left(-c_{13}q_2 k^2 a^2 - c_{12}q_2 n^2 + in^2 ka(c_{13} - c_{12})\right)J_n(s_2 a),$$
$$a_{13} = n(c_{11} - c_{12})[(n-1)J_n(s_3 a) - (s_3 a)J_{n+1}(s_3 a)],$$
$$a_{21} = c_{44}[q_1(s_1{}^2 a^2 - k^2 a^2) + 2ika][nJ_n(s_1 a) - (s_1 a)J_{n+1}(s_1 a)],$$
$$a_{22} = c_{44}[(s_1{}^2 a^2 - k^2 a^2) + 2ikaq_2][nJ_n(s_2 a) - (s_2 a)J_{n+1}(s_2 a)],\ a_{23} = c_{44}(inka)J_n(s_3 a),$$
$$a_{31} = n(c_{11} - c_{12})(1 + ikaq_1)[(1-n)J_n(s_1 a) + (s_1 a)J_{n+1}(s_1 a),$$
$$a_{32} = n(c_{11} - c_{12})(q_2 + ika)[(1-n)J_n(s_2 a) + (s_2 a)J_{n+1}(s_2 a)],$$