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OBTAINING DISPERSION CURVES BY A NOVEL NUMERICAL METHOD

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Abstract
The traditional methods for obtaining frequency equations and plotting dispersion curves are usually based on determination of the zeros of the frequency equation by an iterative find-root algorithm. This approach is very time consuming and prone to numerical errors. In this paper a new method is proposed in which the dispersion curves are extracted from a three-dimensional illustration of the frequency equation. The method starts with plotting of a three-dimensional representation of the frequency equation. Then, by a suitable cut along the phase velocity-frequency plane of this graph, the dispersion curves, which are the numerical solutions of the frequency equation, are obtained. Compared to the traditional methods, this approach is very fast and simple. The possibility of numerical errors is very low and the final results are presented in a very convenient graphical form.

INTRUDUCTION

The problem of wave propagation in circular cylinders was first studied in terms of the general theory of elasticity by Pochhammer [1] in 1876 and independently by Chree [2] in 1889; however, due to complexity of these equations, they could not present numerical solutions. Later, many researchers studied Pochhammer-Chree frequency equations and solved them for special situations, such as very short or very long wavelengths [3,4,5]. Until 1940, no exact solution existed for these equations and the problem was only studied for simple and special cases. Love [6] and Kolsky

[7] were among those who first solved the frequency equations by numerical methods.

To solve the frequency equations, many researchers use iterative techniques, such as linear or quadratic interpolation or extrapolation algorithms, which are very fast on a single root. However, when two roots are in close proximity, for example near the crossing points of longitudinal mode dispersion curves, the function changes sign twice and such schemes are unstable. Alternatively, the frequency equations may be solved by safer iteration techniques such as Newton-Raphson, bisection, and Mueller. However, because of the high variety and number of operations, these methods are difficult, very slow, and time consuming.

In this paper, an alternative method is proposed that is more stable, very fast, with low possibility of numerical error. It produces a convenient graphical illustration of the frequency equation which has no counterpart in traditional methods.

PROPAGATION OF LONGITUDINAL WAVES IN A CIRCULAR CYLINDER

To demonstrate the advantages of the proposed method, the problems of longitudinal wave propagation in isotropic and transversely isotropic cylinders are solved numerically and the three-dimensional representation of the frequency equations as well as the corresponding dispersion curves are plotted.

To demonstrate the capability of the proposed numerical method in solving frequency equations, the problem of longitudinal guided wave propagation in isotropic and transversely isotropic circular cylinders is considered in this section. The method can be easily applied to other problems such as propagation of waves in isotropic and anisotropic plates and shells.

A cylindrical coordinate system, (r,θ,z), is chosen with the z direction coincident with the axis of the cylinder.

The displacement vector is written in terms of three scalar potential functions as follows [8]:

$$U = \nabla\phi + \nabla\times(\chi\hat{e}_z) + a\nabla\times\nabla\times(\psi\hat{e}_z) \tag{1}$$

The scalar potentials for the cylinder are of the form:

$$\phi(r,\theta,z,t) = \sum_{n=0}^{\infty}(B_n J_n(s_1 r) + q_2 C_n J_n(s_2 r))\cos(n\theta)e^{i(kz-\omega t)}\ , \tag{2}$$

$$\psi(r,\theta,z,t) = \sum_{n=0}^{\infty}(q_1 B_n J_n(s_1 r) + C_n J_n(s_2 r))\cos(n\theta)e^{i(kz-\omega t)}\ , \tag{3}$$

$$\chi(r,\theta,z,t) = \sum_{n=0}^{\infty} D_n J_n(s_3 r)\sin(n\theta)e^{i(kz-\omega t)}\ . \tag{4}$$

where $k = \omega / c_p$ is the wave number, c_p is the phase velocity, ω is the circular frequency and J_n are first type Bessel functions of order *n*. Moreover, s_1, s_2, s_3, q_1 and q_2 are constants which depend on the elastic constants of the material as well as the frequency and wave number.

For axisymmetric problems, i.e. longitudinal waves traveling along the cylinder axis, the displacement field is independent of θ and of the form (*ur, 0, uz*). This mode of wave propagation corresponds to *n=0* in Eqs. (2)-(4) and results in $\chi = 0$ [10].

For a cylinder in vacuum, the traction-free boundary conditions hold. Therefore, at *r=a,*

$$\sigma_{rr} = \sigma_{rz} = 0, \tag{5}$$

Substitution of equations of stress, expressed in terms of potential functions, in Eq. (5) gives [9],

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{pmatrix} B_0 \\ C_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{6}$$

The frequency equation, which is also called the dispersion equation, shows the relationship between the frequencies and phase velocities of various modes of longitudinal guided waves in the cylinder; for this problem, it can be expressed as [9],

$$\det[\mathrm{a}_{\mathrm{ij}}] = 0, \tag{7}$$

NOVEL NUMERICAL METHOD

In this section, the new approach for solving the frequency equation is described. In this approach there is no need to search for the zeros of the secular determinant of Eq. (7) in order to plot the dispersion curves. The LHS of Eq. (7) which is in terms of two variables, f and c_p is calculated for the desired range of frequencies and phase velocities. A three-dimensional plot of the real part (arbitrary units) of the frequency equation ($real[\det(a_{ij})]$) versus dimensionless frequency fd and dimensionless velocity c_p / c_b is produced, where c_b is the bar velocity. A two-dimensional cross section of this three-dimensional plot in $fd - c_p / c_b$ plane provides the desired dispersion curves. Despite the simplicity of this method, it is very fast and accurate. In fact, the cumbersome approach of checking individual points one by one is now replaces by looking at the big picture of the solution. By avoiding the one-by-one examination of the points, the calculations are done more quickly, and the whole procedure is less prone to numerical errors.

To show the advantages of the proposed new technique, we use it for plotting the dispersion curves of a number of components. Figure (1) shows the three-dimensional frequency equation plot of longitudinal modes of an aluminum cylinder.

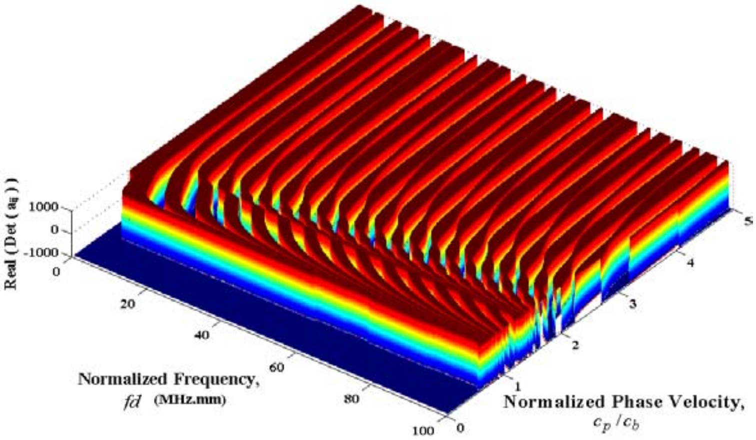

Figure 1- Three-dimensional plot of the frequency equation for an isotropic aluminum cylinder,

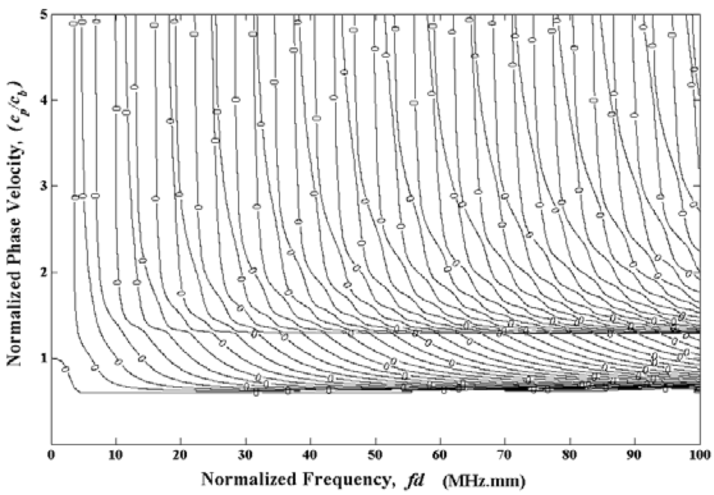

Figure 2- Two-dimensional plot of the frequency equation for an isotropic aluminum cylinder,

The elastic constants of materials were taken from Table I of Ref. 9. Figure (2) is the two-dimensional cross-section of Fig (1) in $fd - c_p / c_b$ plane. The upper and lower limits of $real[\det(a_{ij})]$ are truncated at ± 1000 to enhance the graphical representation of the results. Figures (3) and (4) show the frequency equation and dispersion curves for a transversely isotropic CFAMC composite cylinder;

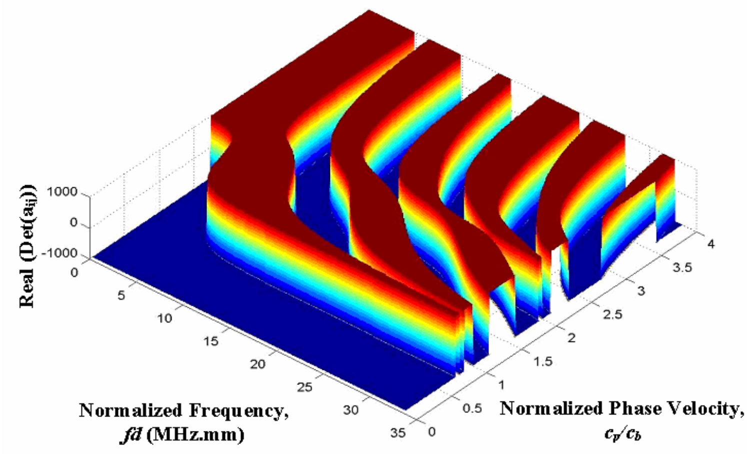

Figure 3- Three-dimensional plot of the frequency equation for a transversely isotropic CFAMC cylinder,

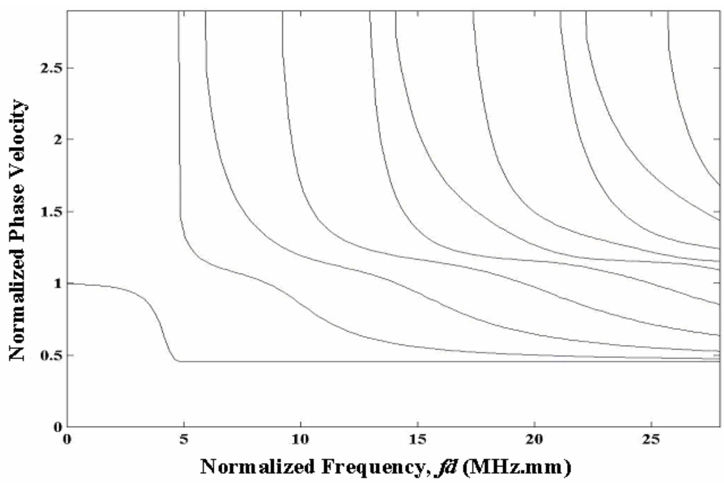

Figure 4- Two-dimensional plot of the frequency equation for a transversely isotropic CFAMC cylinder,

In the examples presented in this section, the procedure was very fast. Figures 2 and 4 were plotted in 4.26 and 2.98 seconds, respectively. The corresponding computation times for plotting Figs. 2 and 4 by iterative techniques were 106.65 and 78.52 seconds. This is significantly faster than calculating and plotting the dispersion curves by traditional iterative techniques. The calculations were performed on a personal computer having a 1.7 GHz Pentium 4 processor with 128 mega bytes of RAM.

Table 1, Physical parameters.

Material	Stiffness $\times 10^{11}$ (N/m^2)					Density (kg/m^3)
	c_{11}	c_{12}	c_{13}	c_{33}	c_{44}	
Aluminum	1.108	0.612	0.612	1.108	0.249	2690
CFAMC	1.79	0.84	0.95	3.09	0.50	3220

CONCLUSIONS

An alternative numerical approach for plotting dispersion curves was presented in this paper. In this method, instead of using repetitive algorithms, a three-dimensional illustration of the frequency equation is used for obtaining the dispersion curves. To show the validity of the proposed approach, it was shown that the results obtained by this method are in full conformity with the results determined by other methods for the isotropic and transversely isotropic circular cylinders. The proposed method can be applied to other problems including wave propagation in isotropic and anisotropic plates and shells. It is very fast, simple, and robust (low possibility of numerical error), and gives a clear picture of the nature of the frequency equation.

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