ACOUSTICAL WAVES GENERATED BY A PULSATING SPHERE WITH TIME-VARYING RADIUS

Lakhdar Gaffour

Department of Physics, Faculty of Sciences,
Sidi-Bel-Abbes University 22000, Algéria
(lgaffour@yahoo.fr)

Abstract
This paper presents a theoretical approach for analyzing the behavior of acoustical waves due to a pulsating sphere with a time- varying radius. Because of the interdependence of the space and time coordinates, the classic analytical methods cannot be applied for solving the governing equation of the system. The problem is solved using an analytical method developed in previous papers. The exact solution is then determined and the analysis of the acoustic behavior shows that all the physical properties of the system are dynamic.

INTRODUCTION

Acoustic waves propagate in a variety of media. The presence of various acoustic waves in the surrounding media produces a continuum of interactive acoustic events which occur throughout the universe at various scales.

The physical dimensions of an acoustic source represent an important parameter which determines the vibrating properties of the corresponding acoustic waves. Normally, an acoustic waves source vibrates at a frequency which is related to the length or diameter of the source, and, of course, depends upon of other parameters.

The propagation of acoustic waves generated by a pulsating sphere was investigated for application in many areas [1-6]: Medicine, Aerospace, Geophysics, etc. The problem of the classical case is well-known that is when the radius of the sphere is a constant. However, if the radius of the sphere is a function of time, the situation becomes very difficult for determining the solution of the problem. In the most cases, the attempts for solving this question have utilized different numerical techniques or general mathematical methods which provide complex solutions. For this reason, we propose an analytical solution for acoustic waves generated by a pulsating sphere with a varying- time radius. The solution is deduced from an analytical method previously published [7, 8]. The solution is obtained in closed form and then makes easy the understanding of the physical phenomenon.

2. POSITION OF THE PROBLEM

For a spherically symmetric acoustic wave in terms of pressure p the governing equation is reduced to the following wave equation:

$$\frac{\partial^2 U(\tau,r)}{\partial \tau^2} - \frac{\partial^2 U(\tau,r)}{\partial r^2} = 0 \ , \tag{2-1}$$

with $U(\tau, r) = r.P(\tau, r)$ and $\tau = ct$, where c, t, r are respectively the wave velocity, time and the sphere radius.
If the radius becomes a function of time $r = a(\tau)$ with $a(0) = a_0$, then the solution of Eq.(2.1) is not obvious. Indeed, because of interdependence of space and time coordinates, the method of separation of variables cannot be applied. Thus, the problem amounts to solve Eq. (2.2) in the following time-varying domain

$$\tau \rangle 0 \ ; \ \ 0 \le r \le a(\tau) \tag{2.2}$$

The initial and boundary conditions are not needed; they vary with the respective applications and are specified there.

3. SOLUTION OF THE PROBLEM

The principle of the method consists in transforming the variable domain ($0 \le r \le a(\tau)$) into a fixed band ($0 \le \eta \le \eta_0$) by using a conformal mapping given by the following analytical function

$$F(W) = \tilde{f}(\xi,\tilde{\eta}) + i\tilde{g}(\xi,\tilde{\eta}). \tag{3.1}$$

By imposing a condition upon F, new real transformations are determined which leave the wave equation invariant in the fixed domain where the solution is possible. The solution in the initial domain is obtained by the inverse function transformations.

Our approach uses the analogy between Laplace's equation and the wave equation. Indeed, the change $\tilde{r} = ir$ ($i^2 = -1$) transforms Eq. (2.1) into the following elliptic equation:

$$\frac{\partial^2 U(\tilde{r},\tau)}{\partial \tau^2} + \frac{\partial^2 U(\tilde{r},\tau)}{\partial \tilde{x}^2} = 0 \tag{3.2}$$

It is well known that solutions of Laplace's equation remain solutions of Laplace's equation when subjected to a conformal transformation. More precisely, if

$$Z = F(W) = \tilde{f}(\xi,\tilde{\eta}) + i\tilde{g}(\xi,\tilde{\eta}) \tag{3.3}$$

where F is an analytical function of the complex variable $W = \xi + i\tilde{\eta}$ with $Z = \tau + i\tilde{r}$, then the change of variables

$$\tau = \tilde{f}(\xi,\tilde{\eta}) \ ; \quad \tilde{r} = \tilde{g}(\xi,\tilde{\eta}), \tag{3.4}$$

transforms (3.2) into the following form:

$$\frac{\partial^2 U(\tilde{\eta},\xi)}{\partial \xi^2} + \frac{\partial^2 U(\tilde{\eta},\xi)}{\partial \tilde{\eta}^2} = 0 \tag{3.5}$$

Next, we impose the following condition upon F:

$$F^*(W) = F(W^*), \tag{3.6}$$

where asterisk sign means the complex conjugate. The functions $\tilde{f}(\xi,\tilde{\eta})$ and $\tilde{g}(\xi,\tilde{\eta})$ are respectively even and odd with respect to the variable $\tilde{\eta}$. By Mac-Laurin expansion we deduce

$$\tilde{f}(\xi,\tilde{\eta})\Big|_{\tilde{\eta}=i\eta} = f(\xi,\eta) \tag{3.7}$$

$$\tilde{g}(\xi,\tilde{\eta})\Big|_{\tilde{\eta}=i\eta} = ig(\xi,\eta) \tag{3.8}$$

where f and g are real functions of the real variables ξ and η. Letting $\tilde{\eta} = i\eta$ in (3.4) and using (3.7) and (3.8) we find the original variables τ and x

$$\tau = f(\xi,\eta); \quad r = g(\xi,\eta) \tag{3.9}$$

From the relations (3.4) which satisfy the Cauchy-Riemann's conditions, we can show that the relations (3.9) satisfy the following fundamental conditions:

$$\frac{\partial\tau}{\partial\xi}=\frac{\partial r}{\partial\eta}, \quad \frac{\partial\tau}{\partial\eta}=\frac{\partial r}{\partial\xi} \tag{3.10}$$

which are unlike Cauchy-Riemann relations, but, play the same role as the Cauchy-Riemann relations in the transformation theory of Laplace's equation.

In addition to the relations (3.10), the transformation functions themselves satisfy wave equation. Indeed, by using the inversion theorem in derivation of (3.10), we obtain:

$$\frac{\partial^2\tau}{\partial\xi^2}=\frac{\partial^2\tau}{\partial\eta^2}, \quad \frac{\partial^2 r}{\partial\xi^2}=\frac{\partial^2 r}{\partial\eta^2} \tag{3.11}$$

These least relations represent wave equations. According to the above results and with the new variables ξ and η, Eq. (3.5) takes the following form:

$$\frac{\partial^2U(\xi,\eta)}{\partial\xi^2}-\frac{\partial^2U(\xi,\eta)}{\partial\eta^2}=0 \tag{3.12}$$

The right choice of F, produces a conformal mapping of the time-varying domain $0\le r\le a(\tau)$ to a band $0\le\eta\le\eta_0$. In this case, the moving boundary is transformed to a fixed boundary.

The solution for $U(\xi,\eta)$ in the fixed domain is well known and can be expressed in terms of complexes Fourier series.

$$U(\xi,\eta)=\sum_{-\infty}^{+\infty}A_n\left\{\exp\tfrac{i\pi n}{\eta_0}(\xi+\eta)-\exp\tfrac{i\pi n}{\eta_0}(\xi-\eta)\right\} \tag{3.13}$$

In order to find the solution with the original variables τ and r, we consider the inverse function ψ of F such that ψ = F^{-1}. Then,

$$\psi(Z)=F^{-1}F(W)=W \tag{3.14}$$

ψ satisfies also the condition

$$\psi^*(Z)=\psi(Z^*) \tag{3.15}$$

then, we obtain the relations between the new and original variables.

$$\xi+\eta=\psi(\tau+r); \tag{3.16}$$

$$\xi - \eta = \psi(\tau - r) \tag{3.17}$$

thus,

$$\xi + \eta_0 = \psi\big(\tau + a(\tau)\big) \tag{3.18}$$

$$\xi - \eta_0 = \psi\big(\tau - a(\tau)\big) \tag{3.19}$$

and the width η_0 of the band can be expressed by:

$$\eta_0 = \tfrac{1}{2}\Big[\psi\big(\tau + a(\tau)\big) - \psi\big(\tau - a(\tau)\big)\Big]. \tag{3.20}$$

Finally, the exact solution with the original variables is given of the form:

$$U(r,\tau) = \sum_{-\infty}^{+\infty} A_n \left\{\exp \tfrac{i\pi n}{\eta_0}\psi(\tau + r) - \exp \tfrac{i\pi n}{\eta_0}\psi(\tau - r)\right\} \tag{3.21}$$

Since $\underline{U}(\tau, r) = r.P(\tau, r)$, then , we deduce the exact expression of the pressure

$$P(r,\tau) = \sum_{-\infty}^{+\infty} \frac{1}{r} A_n \left\{\exp \tfrac{i\pi n}{\eta_0}\psi(\tau + r) - \exp \tfrac{i\pi n}{\eta_0}\psi(\tau - r)\right\} \tag{3.22}$$

RESULTS AND DISCUSSION

In this work, an exact solution for acoustic waves generated by a sphere with time-varying radius has been presented. As indicated in (3.22), the solution is of the same form of the classical case (r constant). However, it expresses in terms of functional Fourier's series.

We note that if the radius is assumed to be constant ($r = a(\tau) = a_0$), the relation (3.22) leads to the well-known classical solution. Indeed, in this case, the function F and its inverse ψ become identity. Thus, this result supports the applicability of the solution (3.22).

Relation (3.22) represents the general solution of the problem. The first term represents a divergent wave propagating in all direction from the origin. The second term represents a convergent wave which approaches to the origin.

For an actual case, we have to know the initial and boundary conditions and also the function ψ. We must point out that anyone of the eigenfunctions of the static sphere (r constant) can be chosen as initial conditions and for any moving law of the radius there is a specific function ψ.

In spite of its general expression, (3.22) shows a modal nature of the pressure. It shows also that every mode is dynamic because of η_0 which depends on time according to (3.20). This means that the frequency is instantaneous. We remark also that the amplitude of the pressure decreases when the radius increases as in classical case. However, we expect that the amplitude is also instantaneous. Therefore, we expect the existence of a generalized double Doppler's shift.

The determination of the pressure under this closed form makes easy the understanding physical phenomenon and the computational of different physical entities such energy etc. In the future, other works should examine actual cases of radius motion.

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