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FAST EVALUATION OF THE RAYLEIGH INTEGRAL AND APPLICATIONS TO INVERSE ACOUSTICS

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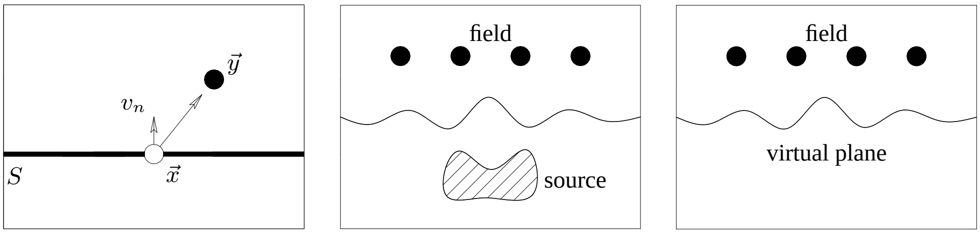
Abstract

In this paper we present a fast evaluation of the Rayleigh integral, which leads to fast and robust solutions in inverse acoustics. The method commonly used to reconstruct acoustic sources on a plane in space is Planar Nearfield Acoustic Holography (PNAH). Some of the most important recent improvements in PNAH address the alleviation of spatial windowing effects that arise due to the application of a Fast Fourier Transform to a finite spatial measurement grid. Although these improvements have led to an increase in the accuracy of the method, errors such as leakage and edge degradation can not be removed completely. Such errors do not occur when numerical models such as the Boundary Element Method (BEM) are used. Moreover, the forward models involved converge to the exact solution as the number of elements tends to infinity. However, the time and computer memory needed to solve these problems up to an acceptable accuracy is large. We present a fast ($\mathcal{O}(n \log n)$ per iteration) and memory efficient ($\mathcal{O}(n)$) solution to the planar acoustic problem by exploiting the fact that the transfer matrix associated with a numerical implementation of the Rayleigh integral is Toeplitz. In this paper we will address both the fundamentals of the method and its application in inverse acoustics. Special attention will be paid to comparison between experimental results from PNAH, IBEM and the proposed method.

INTRODUCTION

Rayleigh's second integral gives the pressure field for given normal velocities on an infinite plane [2]. It can be written as follows (see figure 1(a)):

$$p(\vec{y}) = 2i\omega\rho_0 \int_S G(|\vec{y} - \vec{x}|) v_n(\vec{x}) dS \qquad \text{with} \qquad G(r) = \frac{e^{-\mathrm{i}kr}}{4\pi r} \tag{1}$$



(a) Rayleigh integral (cross section) (b) source and virtual plane (c) reconstruction

Figure 1: *Inverse acoustics using the Rayleigh integral*

Where $\vec{x}$ and $\vec{y}$ denote the source and field points respectively, i, ω, ρ_0 and k are respectively, the imaginary unit, the frequency in radians per second, the density of the fluid at rest and the wave number $= \omega/c_0$, where c_0 is the speed of sound. p, v_n and S denote respectively, the field pressure, the source normal velocity and the set of all points on the infinite plane. If S is has finite area, then the equation gives the pressure field for a source in an infinite baffle – an acoustically hard surface that does not vibrate.

The Rayleigh integral can be used for inverse acoustics as follows. Given a source radiating sound to a *finite* number of sensors, named the field (see figure 1(b)), one can define an infinite virtual plane between the source and the field. One can regard the virtual plane, rather than the physical source, as the radiator of sound. This change of boundary conditions does not change the measurement but reconstruction to the plane is simpler. We now have the situation in figure 1(c). If it is assumed that the sensors are located close to the physical source, then the normal velocities far away from the sensors are negligible. Neglecting these velocities is equivalent to placing a finite virtual plane in an infinite baffle. Hence, a finite virtual plane is used in the reconstruction. The accuracy of the numerical model can be increased by extending the area of the virtual plane and by decreasing the size of the elements. Like most numerical models, this (forward) model converges to the exact solution if the number of degrees of freedom tends to infinity.

Note that in this derivation, the number of sensors is not assumed to be infinite. In NAH techniques, it is necessary to obtain the Fourier transform of the field pressure as a function of the location. Naturally, this Fourier transform can not be found exactly based on a finite number of measurements, and an approximation needs to be made by means of windowing techniques. In the current methods, windowing techniques are not necessary and the forward model converges to the exact solution of the Helmholtz equation irrespective of the number of sensors. Finally, it is noted that calculating a rate of convergence for the *inverse* model as a function of the source area remains difficult but numerical experiments demonstrate that this model converges as well [8].

FORWARD MODEL

In the derivation below, the physical source is not significant. Therefore, the virtual plane will be referred to as the source for reasons of simplicity. In this section we demonstrate the structure of a transfer matrix that follows from the a discretized Rayleigh integral. This property only occurs on regular grids. Hence, we assume that the source grid is equidistant and that a sensor is placed above every node. As demonstrated in a later section, the latter assumption can be weakened. For reasons of simplicity, we consider only constant elements with a single Gauss point here, but the structure has been proven for a much wider range of models. In this simple case, the model becomes:

$$\mathbf{Hv} = \mathbf{p} \quad \text{with} \quad h_{ij} = 2\mathrm{i}\omega\rho_0 S_e G(|\vec{y}_i - \vec{x}_j|) \tag{2}$$

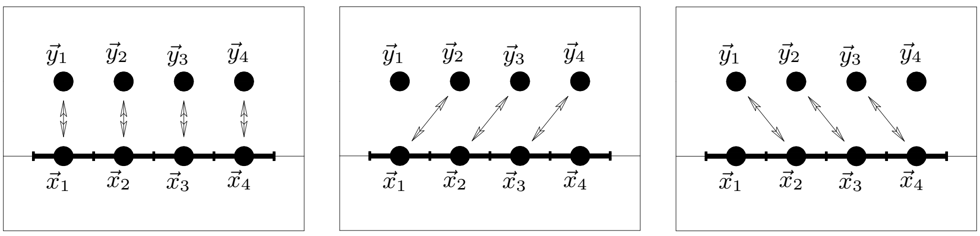
where $\mathbf{H}$, $\mathbf{v}$ and $\mathbf{p}$ are the transfer matrix and the vectors containing source velocities and field pressures respectively. S_e is the area of an element. $\vec{x}_i$ and $\vec{y}_i$ denote the locations of a source Gauss point and a field point respectively.

Each element of the transfer matrix depends on the distance $|\vec{y}_i - \vec{x}_j|$ only: this is the distance between the source point with number j and the field point with number i. The Toeplitz property (see equation 3) is revealed when it is noted that the same distances occur in a very specific structure. From figures 2(a) and 2(b), it can be deduced that the matrix is Toeplitz. Furthermore, from figure 2(c), it can be seen that the matrix is symmetric.

$$\left.\begin{array}{lll} \text{Toeplitz} & : & h_{i-1,j} = h_{i,j+1} \\ \text{symmetric} & : & h_{i,j} = h_{j,i} \end{array}\right\} \qquad \mathbf{H} = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{12} & h_{11} & h_{12} & h_{13} \\ h_{13} & h_{12} & h_{11} & h_{12} \\ h_{14} & h_{13} & h_{12} & h_{11} \end{bmatrix} \tag{3}$$

Due to this structure, the transfer matrix is fully determined by its first column. Hence, the time and memory needed to calculate and store the transfer matrix can be reduced from $\mathcal{O}(n^2)$ to $\mathcal{O}(n)$. Although this property is not abundantly present in literature, it should be noted that it is not a new discovery: Toeplitz matrices have been proven to exist for a wide range of convolution operators on finite grids [1] [8].

The matrix structure is similar for a two-dimensional source. A Toeplitz property can then be identified in both directions. To be exact: when the nodes on the first row are numbered



(a) occurrences of $|\vec{y}_1 - \vec{x}_1|$ (b) occurrences of $|\vec{y}_2 - \vec{x}_1|$ (c) occurrences of $|\vec{y}_1 - \vec{x}_2|$

Figure 2: *Repeated distances for a Rayleigh integral model with constant elements and a single Gauss point*

1 to k and those in the second row are numbered $k+1$ to $2k$ etcetera, such that the source velocity distribution is not a matrix but a vector, then the structure of the transfer matrix is block Toeplitz with Toeplitz blocks (BTTB) [1]. The memory usage and time needed for numerical integration remains $\mathcal{O}(n)$ (where n is the total number of nodes).

The model discussed here has been chosen for its conceptual simplicity but it is not suited to acoustic nearfield reconstruction. In practice, an adaptive quadrature is used to calculate the numerical integrals accurately. Linear elements are used instead of constant elements and it is possible to measure the acoustical particle velocity as well as the pressure. Importantly, the source mesh is finer and larger than the field grid. Although these modifications do not change the concept of the method, it is stressed that they are crucial for its accuracy.

The method presented above will be referred to as TRIM (Toeplitz Rayleigh Integral Method).

REGULARIZATION

There is a wide range of efficient regularization methods available for Toeplitz matrices [1]. However, if the sensors are placed above only some nodes – i.e. if the source mesh is larger and finer than the field grid – then part of the matrix structure is lost and not all efficient methods can be applied. If the number of sensors is very large, a fast ($\mathcal{O}(n \log n)$) matrix-vector multiplication can be used to make the regularization more efficient. In order to exploit the speed of this operation we apply the Krylov-subspace methods that are also popular in IBEM, where the matrix-vector multiplications are performed with the fast algorithm.

It is noted that the term *fast* refers to the asymptotical speed of this operation. Numerical experiments have indicated no increase in speed when the number of field points does not exceed 1000. In this case, it is equally efficient to apply an ordinary matrix-vector multiplication (keeping in mind that only the first column of the transfer matrix needs to be stored in computer memory).

The regularization methods we use are as follows. LSQR regularization is used for most problems. As an alternative regularization method, we solve the least-squares form of Tikhonov regularization explicitly using a least-squares solver similar to LSQR.

CASE STUDY

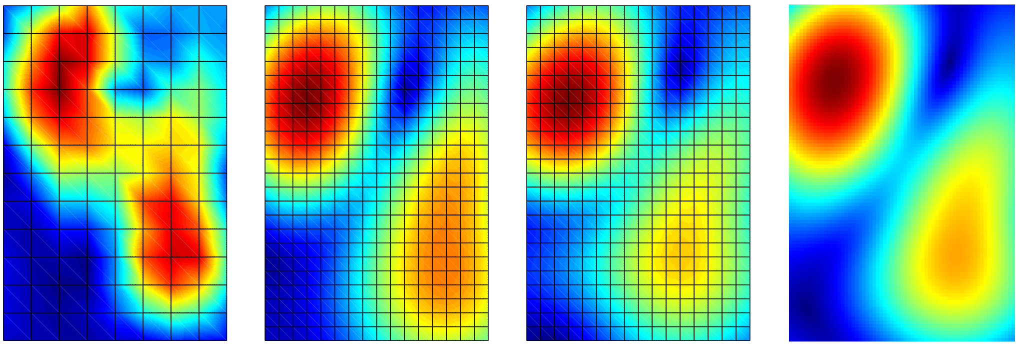
In this section, we compare the TRIM results with results from PNAH, IBEM and a laser vibrometer measurement. The noise source is a hard disk drive suspended in air with wires. The acoustic pressure is measured on an equidistant grid of 17×21 points with a mutual distance of 10 mm. The field grid is 40 mm from the surface of the object. The reader is referred to [3] for more detailed information on the measurement and for details concerning the PNAH technique used.

A reconstruction has been made with an implementation of the boundary element method which has been especially developed for accurate solutions in inverse acoustics [6].

The surface geometry is meshed with linear quadrilateral elements of 5×5 mm, which leads to a total of 1872 elements. Boundary element calculations are performed for several frequencies of interest. The inverse problem is solved with LSQR. No systematic method is used for selecting the regularization parameter. Instead, it is selected using the L-curve, where an attempt was made to select a solution close to the laser data. It is assumed that this solution can not be improved significantly without adding new information and may therefore serve as reference result.

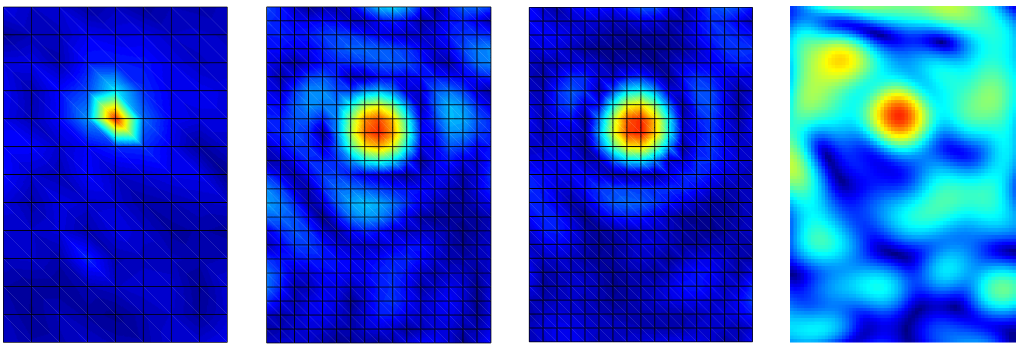
Linear quadrilateral elements of 5×5 mm are used for TRIM and the source is chosen to be extended 0.1m beyond the projection of the measurement grid on all sides. This leads to a total of 5760 elements. LSQR regularization is used and the regularization parameter is chosen in such a way that the differences between IBEM and TRIM are selected such that the results look similar. This way, the observed differences are caused by differences in the method, not by differences in the selected regularization parameter. Finally, it is noted that the PNAH results are selected in the same manner.

The results of the three inverse acoustical techniques are very similar on a frequency of 1075Hz (see figure 3). The PNAH result is slightly closer to the IBEM result than TRIM.



(a) Laser (b) IBEM (c) TRIM (d) PNAH

Figure 3: *Normal velocity (absolute value) at a frequency of 1075Hz*



(a) Laser (b) IBEM (c) TRIM (d) PNAH

Figure 4: *Normal velocity (absolute value) at a frequency of 9668Hz*

This effect may be caused by differences in the regularization methods used. Due to the small difference, it is concluded that all tested methods work well on this frequency.

Significant differences between PNAH and TRIM can be seen on a frequency of 9668Hz (see figure 4). The oscillatory behavior in PNAH is not caused by under-regularization: it also occurs in over-regularized solutions. Since a point-source is present and the number of sensors per acoustic wavelength is low (i.e. 3) it is plausible that the inaccuracy is a sampling artifact. The results of IBEM and TRIM do not have this problem and give similar, accurate, results. This is an indication that TRIM has practical value: the reconstruction of TRIM is accurate on this grid, where a finer grid is probably needed for PNAH.

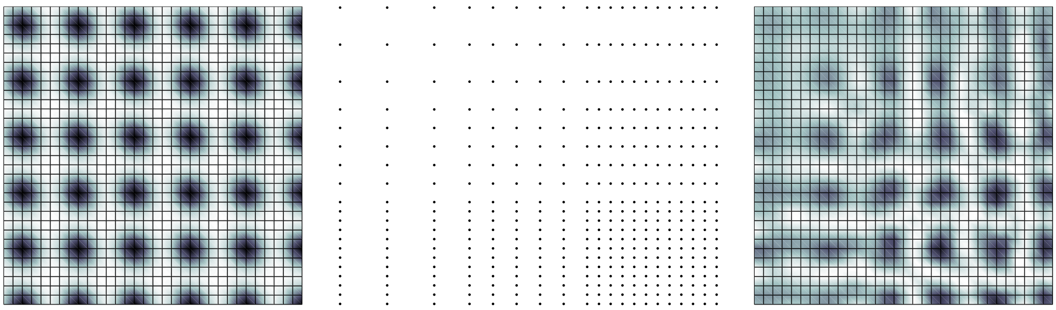
The fact that the results of IBEM and and TRIM are so similar raises a different question. The inverse solver for IBEM has made use of the information that all sound originates from an object with a known geometry. This information is not available to TRIM. It only makes use of the assumption that there the normal velocities are negligible outside the source mesh. However, the results are so similar that we may hypothesize that this lack of information is irrelevant for the current geometry. Theoretical results that quantify this effect lead to new insight and possibly, more general efficient numerical techniques.

Finally, we make a note on the computational cost of TRIM compared to IBEM. The computational times involved in PNAH have not been researched but it is noted that computer resources needed are so small that computational power is not expected to be a problem for PNAH, even for very large data sets. All numerical experiments have been performed on an Intel pentium 4 workstation with a clock speed of 3GHz and 1Gb RAM, running on Windows XP. For IBEM, numerical integration requires approximately 6 minutes per frequency on this very fine mesh. In TRIM, the time needed is between 0.06 and 0.1 seconds, depending on the integration method. The computational cost of regularization has also been studied. For both methods, LSQR regularization is performed for 80 iterations and the time used for IBEM is 2 seconds, versus 3.3 seconds for TRIM. The fact that TRIM is slower than IBEM is presumably caused by the larger number of degrees of freedom in the TRIM model. It is recalled that the asymptotical order of TRIM is lower. The asymptotical order of memory usage is also lower for TRIM because only the first column of the transfer matrix is stored. In practice, the amount of memory used to store the transfer matrix is reduced so much that memory usage is often dominated by the storage of regularization results, which is a problem that can be dealt with.

It may be concluded that TRIM has proven to be a fast, robust and accurate method in the current case study. Although all methods perform well for the frequency of 1050Hz, we have also presented a case where TRIM and BEM give an accurate solution and PNAH does not. Finally, it is noted that the comparison presented here is must not be seen as a summary of an exhaustive survey: such a study has yet to take place.

NUMERICAL EXAMPLE

An advantage of numerical regularization techniques versus FFT-based techniques is their greater flexibility. For example, the current method is not strictly bounded to equidistant grids.



(a) Source distribution (b) Field grid (c) reconstruction

Figure 5: *Numerical example of source identification on a non-equidistant grid*

Although the symmetric Toeplitz structure only exists if a sensor is placed above every source node, it is possible to assign measurement data on only part of this 'virtual' sensor grid and leave the sensor data on the other points undetermined. This means it is possible to have a coarser grid in one area of the field and a finer grid in another. This is illustrated with the following example (see figure 5). A grid of Gaussian distributions is used as source (see figure 5(a)) and the field pressures are calculated at a frequency of 5500Hz, 20mm above the surface, and 5% Gaussian noise is added to the calculated pressures. In the field grid, the distance between the sensors varies spatially (see figure 5(b)). In figure 5(c), a reconstruction is given. It is clearly visible that the result is smoother towards the top left corner of the reconstruction, which is intuitive because there is less information available there. We do not wish to claim that this variation of smoothness reflects the amount of information in any 'optimal' way. We merely state that it varies in a manner which is intuitive.

CONCLUSIONS

An important advantage of PNAH is its conceptual simplicity. The main disadvantage is the fact that a Fourier transform of the spatial data must be obtained from a small number of samples. Even with advanced PNAH techniques, the resulting errors are not always negligible. Also, the padding and windowing techniques needed may be considered less elegant from a theoretical point of view. From a practical point of view, PNAH is an established technique that has proven to give results engineers need.

The forward model of TRIM converges to the exact solution if the element size tends to zero and the source size tends to infinity, which means no windowing techniques are needed. The fact that this method is based firmly in the field of numerical methods leads to several other advantages. First of all, there is a strong link between these methods and the mathematical study of inverse problems. Second, there is a wide range of 'off the shelf' solutions available for common problems such as the automatic selection of a regularization parameter [4] [7]. The main disadvantage is the fact that numerical schemes are needed to obtain a physical model. Although all of these algorithms are well documented [1] [5], they are not yet combined into a commercial package.

Naturally, the main advantage of IBEM is the fact that it is suited to arbitrary geometries and arbitrary distributions of sensors. Disadvantages lie in the computational cost and memory usage which are often a bottleneck for large structures and high frequencies. In the case study, we have seen that IBEM required approximately 6 minutes for integration and regularization on a single frequency where TRIM required 4 seconds. This difference will increase for larger problems.

Finally, it is noted that to the authors knowledge, the current approach can not be generalized to arbitrary geometries, but problems with cylindrical and spherical geometries can possibly result in transfer matrices that have an even more useful structure.

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