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A NEW MEASURE FOR THE NONLINEAR BEHAVIOUR OF PIECE-WISELY LINEAR STRUCTURAL DYNAMIC MODELS

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Abstract

In the ICSV11 prestigious congress, the authors of this paper introduced a criterion to measure the nonlinear behaviour of systems with piece-wisely linear behaviour. In the conclusions section of that paper, it was correctly pointed out that the proposed measure (index) can not consider the effects of the time instants at which the behaviours change. The enhancement of that measure is the objective of this paper. After theoretical discussions and arriving at a new measure, numerical study illustrates the advantages of the new index.

INTRODUCTION

The semi-discretized equation of motion,

$$\begin{array}{lll} \mathbf{M}\ddot{\mathbf{u}}(t)+\mathbf{f}_{\text{int}}=\mathbf{f}(t) & & 0\le t<T, \\ \text{Initial Condition :} & \mathbf{u}(0)=\mathbf{u}_0\,,\quad \dot{\mathbf{u}}(0)=\dot{\mathbf{u}}_0\,,\quad \mathbf{f}_{\text{int}}(0)=\mathbf{f}_{\text{int}_0}\,, & \\ \text{Additional Constraints :} & \mathbf{Q}\,, & \end{array} \tag{1}$$

is the focal point of importance in the analysis of many structural systems. In Eq. (1), $\mathbf{M}$ stands for the mass matrix; $\mathbf{f}_{\text{int}}$ and $\mathbf{f}\,(t)$ respectively express the vectors of internal forces and external excitations; $\mathbf{u}(t)$, $\dot{\mathbf{u}}(t)$, and $\ddot{\mathbf{u}}(t)$ denote the vectors of displacement, velocity, and acceleration, respectively; $\mathbf{u}_0$, $\dot{\mathbf{u}}_0$, and $\mathbf{f}_{\text{int}_0}$ imply the initial status, and $\mathbf{Q}$ represents algebraic constraints, e.g., when involved in impact or elastic-plastic behaviour [5,12], all with respect to the degrees of freedom, set for the system. The true behaviour of all real structural systems is not only dynamic, i.e.,

$$\dot{\mathbf{u}} \neq \mathrm{O}\,, \quad \ddot{\mathbf{u}} \neq \mathrm{O}\,, \quad \dot{\mathbf{f}}(t) \neq \mathrm{O}\,, \tag{2}$$

but also nonlinear, i.e.,

$$\exists\ \mathbf{u}^1\,,\ \ \exists\ p < \infty\ \ :\ \ \mathbf{u}^2 = p\mathbf{u}^1 \quad \not\Rightarrow \quad \mathbf{f}^2 = p\mathbf{u}^1\ \ . \tag{3}$$

Considering practical aspects, this behaviour can in occasions, be simplified to linear dynamic, nonlinear static, and even, linear static behaviours. Nevertheless, there are many situations, where, the nonlinear dynamic behaviour dominates, and especial attention to correct analysis of behaviours is essential. The behaviour of telecommunication towers subjected to severe winds, the behaviour of multi-span cable-stayed bridges subjected to seismic pounding of the decks, and fluid-structure interaction are good examples.

With regard to nonlinear behaviour, since Eq. (3) implies an inequality, the notion of nonlinearity is different in different problems (the notion of linearity is clear and changeless); expressions such as strongly nonlinear, weakly nonlinear, highly nonlinear, etc., are referred in the literature by times [1,6,10]. In order to distinguish these nonlinear behaviours and arrive at the capability of comparing them, it seems a good idea to define a measure (index). This is already provided in other branches of science; see [4,7,9]. In structural engineering, and specifically with regard to Eq. (1), the authors seem to have taken the first steps by introducing the number of the distinct changes of the characteristics as a nonlinearity measure for systems with piece-wisely-linear nonlinear behaviour [11]. The measure proposed in ICSV11 is not only limited to piece-wisely linear behaviours, but also, is not sensitive to the time intervals between the distinct changes. In this paper, in view of the facts below:

1- The behaviour of many practical nonlinear structural systems, especially those involved in more severe nonlinearities is piece-wisely linear, e.g., impact, linearly-elastic/perfectly-plastic behaviours,
2- The numerical procedures analyzing nonlinear behaviours convert nonlinear behaviours to piece-wisely linear behaviours with converging corresponding responses [2,3],

we do not attempt to broaden the study presented in ICSV11. However, to arrive at a better and more reasonable measure for nonlinearity, the time distinction between the changes of characteristics and the severity of nonlinearity at each interval is taken into account. To consider the severity of the nonlinear behaviour at each interval, attention is paid to the sources of nonlinearity at each interval, and meanwhile, the fact that for the special case of piece-wisely-linear nonlinear behaviours,

$$\exists\, \mathbf{u}^L :\ \ -\left|\mathbf{u}^L\right| \leq \mathbf{u} \leq \left|\mathbf{u}^L\right|,\quad \text{linear behaviour}\,({}_0\mathbf{M}\ddot{\mathbf{u}}+{}_0\mathbf{C}\dot{\mathbf{u}}+{}_0\mathbf{K}\mathbf{u}={}_0\mathbf{f})\,, \tag{4}$$

where, by applying $|\ |$ to a vector we mean the vector of the absolute values of all the members of the vector, and by the sign $\le$ between two vectors, we consider the $\le$ sign between the components of the two vectors.

To arrive at an appropriate measure; a new index (measure) of nonlinearity is proposed in the next section; the performance of the proposed index is then studied numerically; and finally, the paper is closed with a brief set of conclusions.

THEORY

Starting from linear dynamic behaviour, consider the equation of motion below,

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f} \,. \tag{5}$$

In Eq. (5), $\mathbf{C}$ and $\mathbf{K}$ respectively imply the damping and stiffness matrices. The behaviour converts from linear to piece-wisely linear, when, at least, at a time instant, abrupt changes occur in one or more of the $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$, $\mathbf{f}$, or $\dot{\mathbf{u}}$; and the corresponding time instants and/or the changes noted above depend on unknowns, e.g., the history of displacement. In more detail, in view of the equation below,

$$\begin{aligned} &{}_i\mathbf{M}\,{}_i\ddot{\mathbf{u}} + {}_i\mathbf{C}\,{}_i\dot{\mathbf{u}} + {}_i\mathbf{K}\,{}_i\mathbf{u} = {}_i\mathbf{f} \qquad {}_it_0 \le t < {}_it_e \,, \\ &{}_i\mathbf{u}(0) = {}_i\mathbf{u}_0 \,, \\ &{}_i\dot{\mathbf{u}}(0) = {}_i\dot{\mathbf{u}}_0 \,, \end{aligned} \tag{6}$$

where, the left subscript is an identifier for the arbitrary interval with linear behaviour within, the restriction,

$${}_it_e = {}_{i+1}t_0 = g(\mathbf{u}, \dot{\mathbf{u}}, \ddot{\mathbf{u}}, \ldots) \tag{7}$$

(where, g schematically implies the dependence of ${}_it_0$ and ${}_it_e$ on the displacements and their time derivatives), together with Eq. (6), and

$$\begin{aligned} \forall\, i: \quad &{}_{i-1}\mathbf{M} \ne {}_i\mathbf{M} \quad or \quad {}_{i-1}\mathbf{C} \ne {}_i\mathbf{C} \quad or \quad {}_{i-1}\mathbf{K} \ne {}_i\mathbf{K} \quad or \\ &{}_{i-1}\mathbf{f}\left({}_{i-1}t_e\right) \ne {}_i\mathbf{f}\left({}_it_0\right) \quad or \\ &{}_{i-1}\dot{\mathbf{u}}\left({}_{i-1}t_e\right) \ne {}_i\dot{\mathbf{u}}\left({}_it_0\right), \end{aligned} \tag{8}$$

define a nonlinear dynamic behaviour of piece-wisely-linear type in the time interval

$$t_0 \le t < T; \quad t_0 = {}_{Min(i)}t_0 \;, \quad T = {}_{Max(i)}t_e \,. \tag{9}$$

With this consideration, it seems reasonable to state the amount of the severity of the nonlinearity within each interval in Eq. (6) by

$$ {}_iS = \mathrm{Max}\left({}_iS_{\mathbf{M}}, {}_iS_{\mathbf{C}}, {}_iS_{\mathbf{K}}, {}_iS_{\mathbf{f}}, {}_iS_{\dot{\mathbf{u}}}\right), \tag{10} $$

where,

$$ {}_iS_{\mathbf{M}} = \frac{\left\| {}_i\mathbf{M} - {}_{i-1}\mathbf{M} \right\|}{Max\left(\left\| {}_i\mathbf{M} \right\|, \left\| {}_{i-1}\mathbf{M} \right\|\right)}, {}_iS_{\mathbf{C}} = \frac{\left\| {}_i\mathbf{C} - {}_{i-1}\mathbf{C} \right\|}{Max\left(\left\| {}_i\mathbf{C} \right\|, \left\| {}_{i-1}\mathbf{C} \right\|\right)}, {}_iS_{\mathbf{K}} = \frac{\left\| {}_i\mathbf{K} - {}_{i-1}\mathbf{K} \right\|}{Max\left(\left\| {}_i\mathbf{K} \right\|, \left\| {}_{i-1}\mathbf{K} \right\|\right)}; $$

$$ {}_iS_{\mathbf{f}} = \frac{\left\| {}_i\mathbf{f} - {}_{i-1}\mathbf{f} \right\|}{Max\left(\left\| {}_i\mathbf{f} \right\|, \left\| {}_{i-1}\mathbf{f} \right\|\right)}, \tag{11} $$

$$ {}_iS_{\dot{\mathbf{u}}} = \frac{\left\| {}_i\dot{\mathbf{u}}_0 - {}_{i-1}\dot{\mathbf{u}}_0 \right\|}{Max\left(\left\| {}_i\dot{\mathbf{u}}_0 \right\|, \left\| {}_{i-1}\dot{\mathbf{u}}_0 \right\|\right)}, $$

($\| \; \|$ stands for an arbitrary norm [8]). Based on Eqs. (10) and (11), we are now in the stage to introduce the computationally inexpensive index,

$$ Z = \frac{1}{\left({}_kt_e - {}_jt_0\right)} \sum_{i=j}^{i=k} {}_iS\left({}_kt_e - {}_it_0\right), \tag{12} $$

for measuring the nonlinear behaviour at the interval between ${}_jt_0$ and ${}_kt_e$; $k \geq j$.

The performance of the measure proposed above is, in the next section, numerically studied and compared with its ancestor [11] via simple examples.

ILLUSTRATIVE EXAMPLE

As the first example, we would rather consider the example, in view of which, nonlinearity measure is first studied for structural dynamic problems [11], i.e.,

$$ \begin{aligned} & \ddot{u} + u = 0 \qquad 0 \leq t < 30 \\ & u(t=0) = d \\ & \dot{u}(t=0) = 0 \\ & \text{Elastic Contact at } u = -4. \end{aligned} \tag{13} $$

Denoting the number of current collisions with $\overline{n}$, the analytical solution is given by

$$ \begin{aligned} & d \leq 4 \qquad u = d\cos(t) \\ & d > 4 \qquad u = d\cos(t) \qquad 0 \leq t < c \\ & \qquad\qquad u = d\cos\left(t + 2\overline{n}\arccos\left(\frac{4}{d}\right)\right) \qquad (2\overline{n}-1)\,c \leq t < (2\overline{n}+1)\,c, \end{aligned} \tag{14} $$

and,

$$c = \pi - \arccos\left(\frac{4}{d}\right). \tag{15}$$

For

$$d = 1,2,4,8,16,32\,, \tag{16}$$

the displacement history is depicted in Figs. 1(a-f). The values of the nonlinearity indices according to the authors' suggestion in ICSV11 and this paper are, at the top of the figures, reported respectively as Z_{11} and Z_{13}. By comparing the values of Z_{11} and Z_{13}, we can conclude the similar trend of the new and the recently proposed indices (similar to the previous index, the new index takes into account the changes of behaviours).

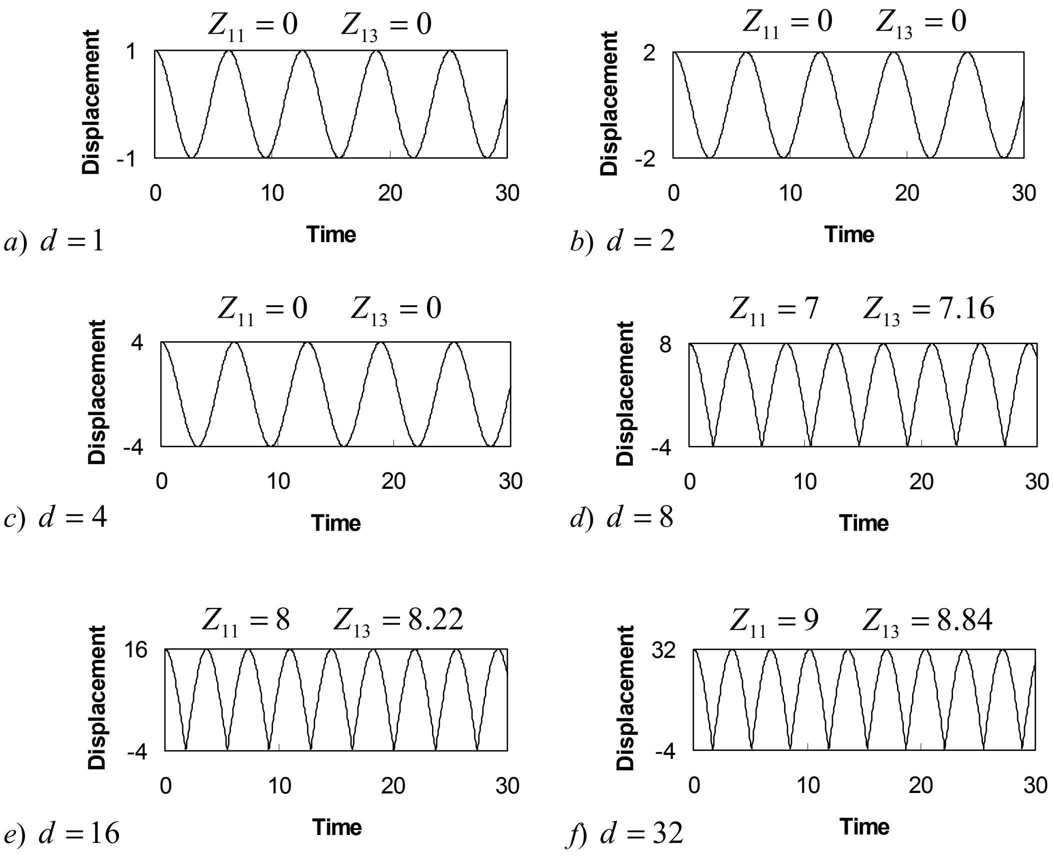

Figure 1 – Time history of displacement for different values of d in Eq. (13), together with the corresponding values of Z_{11} and Z_{13}

To illustrate the advantage of the new index compared to the index proposed at ICSV11 [11], the problem in Eq. (13) is slightly changed by extending the time interval for another 30 seconds and omitting the elastic stop within the extension, i.e.,

$$\begin{aligned}
&\ddot{u} + u = 0 \qquad 0 \le t < 60, \\
&u(t = 0) = d, \\
&\dot{u}(t = 0) = 0, \\
&\text{Elastic Contact at } u = -4 \text{ for } 0 \le t < 30.
\end{aligned} \tag{17}$$

The difference between Eqs. (13) and (17) results in Figs. 2 instead of Figs. 1 (each illustration in Figs. 1 is indeed identical to the first 30 seconds of the corresponding illustration in Figs. 2). The dashed curves in Figs. 2 display the solution of Eq. (17), when the sources of nonlinearity are all omitted. Comparing the first half of each illustration in Figs. 2 with the same illustration in the total interval clearly explains the superiority of the new index to the nonlinearity index proposed at [11]. In brief,

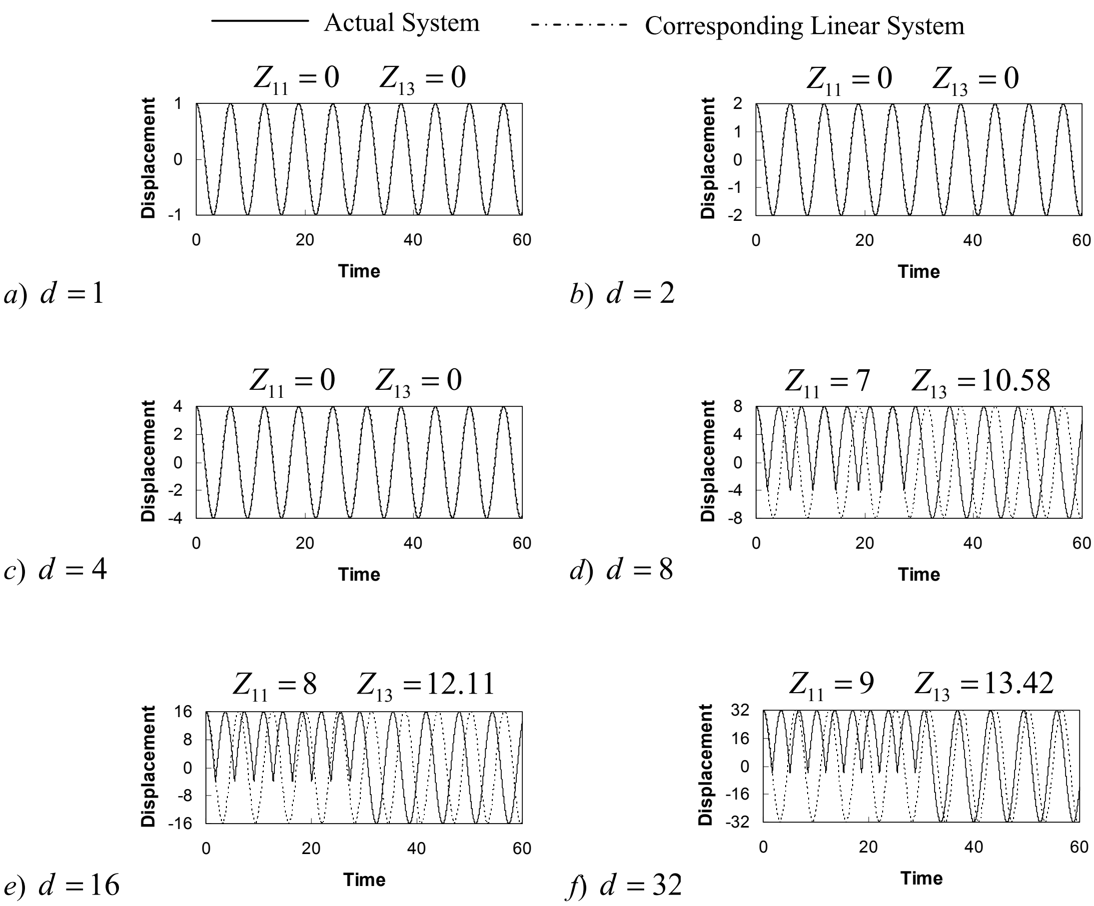

Figure 2 – Time history of displacement for different values of d in Eq. (17), together with the corresponding values of Z_{11} and Z_{13}

regardless of the number of changes of the characteristics, the already existing index [11] can not consider the temporal locations of these changes and hence the values of Z_{11} in the corresponding illustrations in Figs. 1 and 2 are identical. This is not the case for the new index; the corresponding values of Z_{13} in Figs. 2 are reasonably more than those presented by Z_{11}.

CONCLUSIONS

In this paper a new index for measuring the nonlinear behaviour of piece-wisely-linear dynamic systems is proposed. Compared to its ancestor, the new index considers the effects of nonlinear behaviour on the future time instants, in a more accurate fashion. Further research on the subject, regarding real problems of structural dynamics, different types of nonlinearity, etc., is being recommended.

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