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SOUND PROPAGATION OVER ARBITRARY SURFACES WITH VARYING IMPEDANCE

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Abstract

A method is given for evaluating acoustic scattering by an arbitrary irregular surface with spatially varying impedance. This allows examination of the effects of impedance variation and the resulting modification of scattered field. Expressions are derived for the scattered field itself, and for the mean field when the impedance varies randomly.

INTRODUCTION

Many problems in acoustic propagation over irregular terrain are complicated by the presence of impedance variation, and it is often important to identify the scattering effects due to this, or to characterise the surface properties from the scattered field We aim here to provide an efficient means to calculate such effects, and where possible to derive simple descriptions of the resulting coherent or mean field. In order to do this an operator expansion is used: Surface pressures (from which scattered fields are determined) are expressed as the solution of an integral equation, in which the influence of impedance variation is separated. The solution is written in terms of the inverse of the governing integral operator, and provided impedance variation is not too large this inversion can be expanded about the leading term. This provides convenient semi-analytical expressions for scattered fields and equivalent effective boundary conditions for the coherent fields. It also allows efficient evaluation of the solution in cases which would otherwise be computationally intractable, especially for low grazing angles.

The method and results will be described for 2-dimensional problems, but the extension to the 3-dimensional case is straightforward.

GOVERNING EQUATIONS

Consider the acoustic field above an irregular surface with varying impedance in a 2-dimensional medium, with coordinates (x, z) where x is the horizontal and z the vertical, directed upwards. The incident field ψ is assumed to be time-harmonic, with time-dependence $\exp(-i\omega t)$, say, which is henceforth suppressed. Denote the surface profile by $\zeta(x)$, with spatially varying impedance $Z(x) = Z_0 + Z_r(x)$ where Z_0 is a constant reference value. (Where ensemble averages are taken it will be assumed that Z_r is continuous, statistically stationary in x, and has mean zero.) It will also be assumed that Z_r is not large compared with Z_0, in the sense that the root mean square (r.m.s.) of its modulus is less than $|Z_0|$. Everywhere below, angled brackets $<\,\mathbf{.}\,>$ (or for compactness an overbar) denotes ensemble averages with respect to impedance variation.

The field ψ above the surface obeys the Helmholtz wave equation $(\nabla^2 + k^2)\psi = 0$ where k is the wavenumber. Denote by G the free space Green's function, so that (in the 2-dimensional case) G is the zero order Hankel function of the first kind,

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4i} H_0^{(1)}(k|\mathbf{r} - \mathbf{r}'|). \tag{1}$$

The total field ψ along the surface is then given by the solution of a Helmholtz integral equation:

$$\psi_{inc}(\mathbf{r}_s) \;=\; \frac{1}{2}\psi(\mathbf{r}_s) \;-\; \int_{z=\zeta(x)} \left[\frac{\partial G(\mathbf{r}_s, \mathbf{r}')}{\partial n} + \frac{ik_0 G(\mathbf{r}_s, \mathbf{r}')}{Z_0 + Z_r(x')}\right] \psi(\mathbf{r}')dS'. \tag{2}$$

where $\mathbf{r}_s$ here is an arbitrary surface point $(x, \zeta(x))$, and $\mathbf{r}' = (x', \zeta(x'))$. Elsewhere in the upper half space the field can be written as a boundary integral:

$$\psi(\mathbf{r}) \;=\; \int_{z=\zeta(x)} \left[\frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n} + \frac{ik_0 G(\mathbf{r}, \mathbf{r}')}{Z_0 + Z_r(x')}\right] \psi(\mathbf{r}')dS'. \tag{3}$$

where now $\mathbf{r} = (x, z)$ represents a general point in the upper medium. (The right-hand-side of equation (2) is an operator from functions on the real line to itself, and the same holds for (3) if $\mathbf{r}$ is, for example, restricted to a line at fixed z parallel to x.)

SOLUTIONS FOR VARYING IMPEDANCE

Flat surfaces

We first consider the simpler case of a field incident on a varying impedance surface when the surface is flat, say at $z = 0$. This has been studied by many authors in various parameter regimes and is included here simply in order to introduce the technique to be generalised later to rough surfaces.

Noting that

$$\frac{1}{Z_0 + Z_r} \equiv \frac{1}{Z_0} - \frac{Z_r}{Z_0(Z_0 + Z_r)}, \tag{4}$$

equation (2) can be written

$$\psi_{inc}(\mathbf{r}) = (\mathcal{A}_0 + \mathcal{A}_1)\psi(\mathbf{r}), \tag{5}$$

where $\mathbf{r}$ lies on the surface, $\mathcal{A}_0$ is defined by

$$\mathcal{A}_0(\centerdot) = \frac{1}{2}(\centerdot) - \int_{z=0} \left[\frac{\partial G(\mathbf{r},\mathbf{r'})}{\partial z} + \frac{ik_0 G(\mathbf{r},\mathbf{r'})}{Z_0} \right] (\centerdot) dS' \tag{6}$$

and $\mathcal{A}_1$ is given by

$$\mathcal{A}_1(\centerdot) = \frac{ik_0}{Z_0} \int_{z=0} \frac{Z_r(x')G(\mathbf{r},\mathbf{r'})}{Z_0 + Z_r(x')} (\centerdot) dS'. \tag{7}$$

The solution to (5) represents the total field $z = 0$; from this the field elsewhere can be obtained by writing $\psi(\mathbf{r})$ as a superposition of plane waves without recourse to the integral (3).

Suppose for the moment that the impedance is constant, $Z = Z_0$, so that $\mathcal{A}_1$ vanishes. For an incident plane wave, say $\psi_\theta(x,z) = \exp(ik[\sin\theta x - \cos\theta z])$ at an angle θ with respect to the normal, the solution is explicitly

$$\mathcal{A}_0^{-1}\psi_\theta(x,0) \;=\; [1 + R(\alpha)]\exp(i\alpha x), \tag{8}$$

where $\alpha = k\sin\theta$, $\beta = \sqrt{k^2 - \alpha^2}$, and R is the reflection coefficient

$$R(\alpha) = \frac{\beta Z_0 - k_0}{\beta Z_0 + k_0}. \tag{9}$$

Thus $\mathcal{A}_0^{-1} f$ can be found for *arbitrary* $f(x)$ by expressing f as a superposition of plane waves and applying (8). If impedance variation $Z(x) = Z_0 + Z_r(x)$ is now reintroduced, then (5) has formal solution

$$\psi(\mathbf{r}) = (\mathcal{A}_0 + \mathcal{A}_1)^{-1}\psi_{inc}(\mathbf{r}) \tag{10}$$

which may be used to examine the effect of the varying impedance. The inverse can be expanded to give

$$(\mathcal{A}_0 + \mathcal{A}_1)^{-1} \equiv \mathcal{A}_0{}^{-1} - (\mathcal{A}_0{}^{-1}\mathcal{A}_1)\mathcal{A}_0{}^{-1} + (\mathcal{A}_0{}^{-1}\mathcal{A}_1)^2\mathcal{A}_0{}^{-1} - \ldots \tag{11}$$

and as by assumption the effect of the term $\mathcal{A}_1$ is not large, the series converges and the resulting equation can be truncated to obtain an approximation to $\psi(\mathbf{r})$:

$$\psi(\mathbf{r}) \cong \mathcal{A}_0^{-1}\psi_{inc}(\mathbf{r}) \;-\; \mathcal{A}_0^{-1}\left[\mathcal{A}_1\mathcal{A}_0^{-1}\psi_{inc}(\mathbf{r})\right]. \tag{12}$$

The first term on the right of equation (12) is the known specular reflection from a constant impedance surface at $z = 0$; the second models its diffuse modification due to Z_r, i.e. diffraction effects.

Specifically, from (8) and (7) we obtain

$$\mathcal{A}_1(\mathcal{A}_0^{-1}\psi_{inc}(\mathbf{r})) = \frac{ik_0(1 + R(\alpha))}{Z_0} \int_{z=0} \frac{Z_r(x')G(\mathbf{r},\mathbf{r'})}{Z_0 + Z_r(x')} e^{i\alpha x'} dS'. \tag{13}$$

As $\mathcal{A}_0^{-1}$ represents reflection by constant impedance, (13) can be thought of as a secondary 'driving field' for the diffuse term in (12). This field consists of a set of plane waves determined by the Fourier transform of the integral in (13). An example is shown in Figure 1 comparing this term in (12) with the diffuse part of the exact numerical solution. Z_r here has an rms value of around $Z_0/4$. (Note that the incident field has been tapered to minimise edge-effects.)

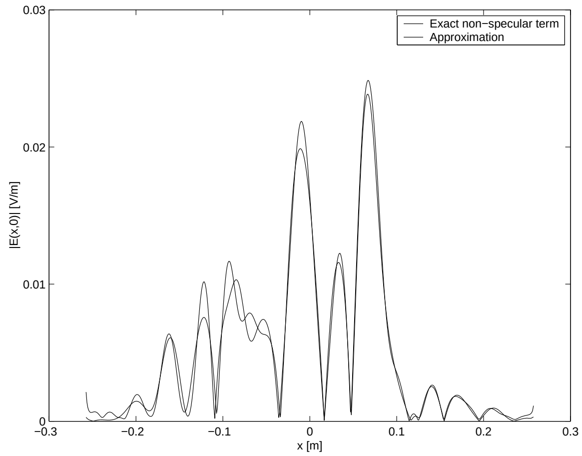

Figure 1: *Comparison of exact and approximate solutions for non-specular component of scattered field for a flat surface with varying impedance.*

As the solution of $\mathcal{A}_0^{-1}$ is known the expression (12) can be evaluated directly, for one or many realisations, and avoids a potentially expensive numerical inversion.

Irregular surfaces

The results above are now extended to obtain expressions for the effect of impedance variation on scattering by an arbitrarily rough surface. This situation is considerably more complicated because, having obtained from (2) the field along the irregular boundary, the scattered field must be obtained by further application of the boundary integral (3) involving variation in both surface profile and impedance.

For a rough surface $z = \zeta(x)$ with impedance $Z = Z_0 + Z_r(x)$ integral equation (2) becomes

$$\psi_{inc}(\mathbf{r}) = (\mathcal{C}_0 + \mathcal{C}_1)\psi(\mathbf{r}), \tag{14}$$

where

$$\mathcal{C}_0(\centerdot) = \frac{1}{2}(\centerdot)\ -\ \int_{z=\zeta(x)} \left[\frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n} + \frac{ik_0 G(\mathbf{r}, \mathbf{r}')}{Z_0} \right] (\centerdot) dS'. \tag{15}$$

and $\mathcal{C}_1$ contains the dependence on Z_r,

$$\mathcal{C}_1(\centerdot) = \frac{ik_0}{Z_0} \int_{z=\zeta(x)} \frac{Z_r(x')G(\mathbf{r}, \mathbf{r}')}{Z_0 + Z_r(x')} (\centerdot) dS'. \tag{16}$$

Even when the impedance is constant, so that $\mathcal{C}_1 = 0$, there is no longer a closed-form analytical solution and in general $\mathcal{C}_0^{-1}\psi_{inc}(\mathbf{r})$ must be evaluated numerically.

Applying the expansion analogous to (11) to $(\mathcal{C}_0 + \mathcal{C}_1)^{-1}$ and truncating gives the approximation for the surface field

$$\psi(\mathbf{r}) \cong \mathcal{C}_0^{-1}\psi_{inc}(\mathbf{r}) - \mathcal{C}_0^{-1} \left[\mathcal{C}_1 \mathcal{C}_0^{-1} \psi_{inc}(\mathbf{r}) \right]. \tag{17}$$

The second term on the right again accounts for the effect of impedance variation Z_r and its interaction with the irregular surface, and the resulting diffraction. The first term corresponds

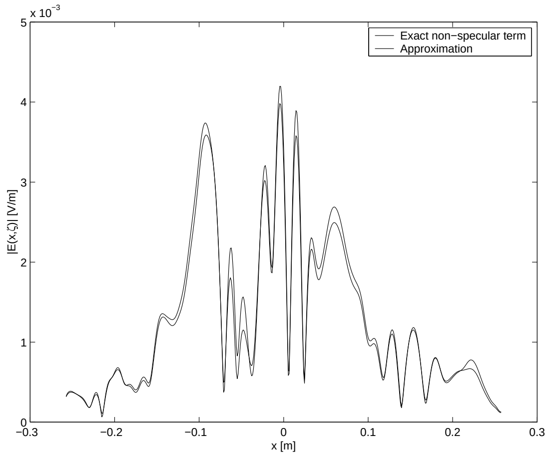

Figure 2: *Comparison of exact numerical solution and approximation for scattered field on rough profile $z = \zeta$ with varying impedance $Z = Z_0 + Z_r$.*

to constant impedance; once this has been obtained the second term is evaluated by applying $\mathcal{C}_1$ and solving again for $\mathcal{C}_0^{-1}$, with the term in square brackets acting as a new driving field. When full numerical inversion of $\mathcal{C}_0$ is necessary, this offers no immediate computational advantage for a single realisation of $\zeta(x)$ and $Z(x)$. However, there are important special cases including low grazing angles for which highly efficient methods are available which cannot be applied directly to the full integral equation (14). Figure 2 compares the surface field term $-\mathcal{C}_0^{-1}\mathcal{C}_1\mathcal{C}_0^{-1}\psi_{inc}(\mathbf{r})$ with the corresponding component of the 'exact' numerical solution of (14). Here the angle of incidence is around 5^o, the ratio of r.m.s. surface height to wavelength $< \zeta^2 >^{1/2} /\lambda = 2/3$, and the ratio of r.m.s. impedance variation to reference value Z_0 is around 1/6. Agreement is again seen to be very close.

This formulation conveniently describes the balance between scattering mechanisms, and is efficient for calculation of the field statistics with respect to impedance variation, as well as allowing theoretical estimates of the field statistics to be obtained.

Mean field

As the impedance is more often known statistically than individually, evaluation of the mean field is important. For flat surfaces the mean field with respect to an ensemble of impedance realisations obeys an effective impedance condition. Thus for an incident plane wave the mean scattered field is specular, but with an 'effective reflection coefficient' depending on incident angle. For an arbitrary surface profile, the coherent field is no longer specular, and its description is therefore more complex. This raises a natural question: is each scattered angle modified by an effective reflection coefficient which is independent of the surface profile? This is found not to be the case.

At a point $\mathbf{r}$ along a given horizontal line above the surface, the field is related to the surface values via the integral (3). If the integral operator here, say $\mathcal{C}'$, is split as before into its constant and varying impedance parts $\mathcal{C}'_0$ and $\mathcal{C}'_1$, then using (17), (3) can be written

$$\begin{aligned}
\psi &= \mathcal{C}'\mathcal{C}^{-1}\psi_{inc} \\
&= (\mathcal{C}'_0 + \mathcal{C}'_1)(\mathcal{C}_0 + \mathcal{C}_1)^{-1}\psi_{inc} \\
&\cong (\mathcal{C}'_0 + \mathcal{C}'_1)(\mathcal{C}_0^{-1} - \mathcal{C}_0^{-1}\mathcal{C}_1\mathcal{C}_0^{-1})\psi_{inc} \\
&\cong \left[\mathcal{C}'_0\mathcal{C}_0^{-1} - \mathcal{C}'_0\mathcal{C}_0^{-1}\mathcal{C}_1\mathcal{C}_0^{-1} + \mathcal{C}'_1\mathcal{C}_0^{-1})\right]\psi_{inc} \qquad (18)
\end{aligned}$$

where we have neglected a term of higher order in $\mathcal{C}_1$. We can now take an ensemble average of (18) with respect to impedance variation, to get the mean modification by impedance variation of the scattered fields.

$$<\psi> \cong \left[\mathcal{C}'_0\mathcal{C}_0^{-1} \;+\; \overline{\mathcal{C}'_1}\mathcal{C}_0^{-1} \;-\; \mathcal{C}'_0\mathcal{C}_0^{-1}\overline{\mathcal{C}_1}\mathcal{C}_0^{-1})\right]\psi_{inc} \qquad (19)$$

where $\overline{\mathcal{C}_1}$ denotes the mean $<\mathcal{C}_1>$, $\overline{\mathcal{C}'_1}$ is defined similarly. The expression (19) can be regarded as semi-analytical: it gives the mean field for an arbitrary irregular surface with randomly varying impedance, but as $\overline{\mathcal{C}_1}$, $\overline{\mathcal{C}'_1}$ depend on the surface profile $\zeta(x)$, numerical evaluation cannot be avoided. This gives rise to a coherent field spectrum with effective coefficients depending on the surface profile. The approximation (19) is equivalent to the solution $\tilde{\psi}$, say, for scattering by the surface $\zeta(x)$ but with constant effective impedance Z_e. This is easily seen by formulating this equivalent problem in terms of integral operators where it becomes

$$\tilde{\psi} \;=\; (\mathcal{C}'_0 + \overline{\mathcal{C}'_1})(\mathcal{C}_0 + \overline{\mathcal{C}_1})^{-1}\psi_{inc}, \qquad (20)$$

and then solving as before and comparing terms with (19).

SUMMARY

Acoustic scattering by an irregular surface with random spatially varying impedance has ben considered. We have sought an efficient method for calculating the field while allowing convenient estimation of the effects of impedance variation and its interaction with the surface profile. The expressions obtained also provide estimates of the mean field with respect to impedance variation. For rough surfaces these are semi-analytical in the sense that numerical evaluation of integrals is needed. (In the case of a flat surface, for which the coherent field is specular, this takes the form of an effective impedance; this is also approximately true for a given irregular surface, but the behaviour is more complicated because of the non-specular nature of the scattered field.) Although for simplicity we have restricted attention to 2-dimensional problems, treatment for 3-dimensional scattering is almost identical.

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REFERENCES

K Attenborough, Acoustic impedance models for outdoor ground surfaces, *J Sound Vib* **99** (1985) 521-544.

S N Chandler-Wilde, S Langdon & L Ritter, A high-wavenumber boundary-element method for an acoustic scattering problem, *Phil Trans R Soc A* **362** (2004) 647-671.

P Boulanger, K Attenborough, S Taherzadeh, T WatersFuller, & K M Li, Ground effect over hard rough surfaces *J Acoust Soc Am* **104** (1998) 1474-1482.

H Giovannini, M Saillard, A Sentenac, Numerical study of scattering from rough inhomogeneous films *J Opt Soc Am A* **15** (1998) 1182-1191.

J A Ogilvy, 1992. Theory of wave scattering from random rough surfaces, *IOP Publishing*.

A.G. Voronovich, *Wave Scattering from Rough Surfaces* (Springer-Verlag, Berlin 1994).

J G Watson & J B Keller, Reflection, scattering, and absorption of acoustic waves by rough surfaces, *Journal of the Acoustical Society of America* **74** (1983) 1887-1894.

D A Kapp & G S Brown, A new numerical method for rough surface scattering calculations, *IEEE Trans. Ant. Prop.* **44** (1996) 711-721.

M Spivack, Moments and angular spectrum for rough-surface scattering at grazing incidence, *Journal of the Acoustical Society of America* **97** (1995) (2), 745-753.

M Spivack, A Keen, J Ogilvy,& C Sillence, Validation of left-right method for scattering by a rough surface, *Journal of Modern Optics* **48** (2001), 1021-1033.