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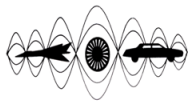

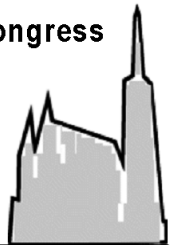

FAST BOUNDARY INTEGRAL SOLUTION FOR ACOUSTIC SCATTERING BY LARGE OBJECTS

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Abstract

The left-right splitting method is developed and applied to problems of acoustic scattering in three dimensions. The boundary integral which governs the solution via the Helmholtz equation can be split into two at each point of observation, corresponding to right- and left-going energy. This leads to a series solution, which, under the assumption that forward scattering is dominant, can often be truncated after one or two terms even in strong scattering regimes. The key advantages are (a) each term can be evaluated very quickly with low memory requirement; and (b) the terms have a straightforward physical interpretation, which gives insight into scattering mechanisms.

INTRODUCTION

The calculation of acoustic scattering by large objects or extended rough surfaces is a challenging computational problem, especially in the presence of strong multiple scattering (e.g. [1, 2, 3]). This is particularly acute at low grazing angles. Boundary integral methods are flexible and often used for such problems but can be computationally intensive and scale badly with increasing frequency.

Here we develop the left-right splitting method and apply this to several acoustic scattering problems. In this approach the unknown surface field is first expressed as the solution to the Helmholtz integral equation, where integration is taken over the surface. The integral can be divided into two at each point of observation, corresponding to right- and left-going energy. This leads to a series solution, which, under the assumption that forward scattering is dominant, can often be truncated after one or two terms even in strong scattering regimes. The key advantages are that (a) each term can be evaluated very quickly with low memory requirement; and (b) the terms have a straightforward physical interpretation, which gives insight into scattering mechanisms. The approach is related to methods recently developed for

electromagnetic scattering in 2-d and 3-d by several authors, eg [4, 5, 7, 8]. In some sense this is motivated by the parabolic integral equation approach [9].

Results have been validated by comparison with "exact" numerical solution where possible, and against analytical results, and found to be robust and convergent. The method can be used to derive scattering results for otherwise intractable cases. The algorithm is well-suited to parallelisation, in which case the speed scales approximately linearly with the number of processors.

In §2 the governing equations and left-right splitting approximation are formulated. The numerical details and main results are shown in §3.

GOVERNING EQUATIONS AND LEFT-RIGHT SPLITTING

Suppose we have a sound source in a 3-dimensional half-space Ω with horizontal axes x, y and vertical axis z directed upwards, bounded by a 2-dimensional rough surface $S = s(x, y)$ varying about the plane $z = 0$. S is assumed continuous and differentiable as a function of x, y. (Exterior scattering problems and arbitrary boundaries can be treated similarly.) A time-harmonic acoustic wave Φ, with wavenumber k obeying the wave equation $(\nabla^2 + k^2)\Phi = 0$ in the region $z > s(x, y)$, results from an incident wave Φ_{inc} at a small grazing angle θ to the horizontal plane. The axes can be chosen so that the principal direction of propagation is at a small angle to the (x, z) plane. We assume that the surface is acoustically hard so that the field obeys the Neumann boundary condition; derivation for pressure release (Dirichlet) surface is similar. Thus $\partial\Phi/\partial\mathbf{n} = 0$, where $\mathbf{n}$ is the outward normal, and G is the free space Green's function $G(\mathbf{r}, \mathbf{r}') = \exp\{ik|\mathbf{r} - \mathbf{r}'|\}/4\pi|\mathbf{r} - \mathbf{r}'|$. The field at a point $\mathbf{r}$ in the medium determined by the surface field via the boundary integral

$$\Phi_{inc}(\mathbf{r}) = \Phi(\mathbf{r}) - \int_S \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n}\Phi(\mathbf{r}')d\mathbf{r}' \tag{1}$$

where $\mathbf{r} = (x, y, z)$ and $\mathbf{r}' = (x', y', s(x', y'))$, say, so that taking the limit as $\mathbf{r} \to \mathbf{r}_s$ gives

$$\Phi_{inc}(\mathbf{r}_s) = \Phi(\mathbf{r}_s) - \int_S \frac{\partial G(\mathbf{r}_s, \mathbf{r}')}{\partial n}\Phi(\mathbf{r}')d\mathbf{r}' \tag{2}$$

where $\mathbf{r}_s = (x, y, s(x, y))$. This integral is interpreted as the limit of the integral in eq. (1) as $\mathbf{r} \to \mathbf{r}_s$ because of the singularity at $\mathbf{r}' = \mathbf{r}_s$. We can write the integration with respect to x,y, and in doing so introduce an additional factor γ into the integrand, where

$$\gamma(\mathbf{r}') = \sqrt{1 + \left(\frac{\partial s}{\partial x'}\right)^2 + \left(\frac{\partial s}{\partial y'}\right)^2}. \tag{3}$$

This can be solved using a method analogous to that applied to the electromagnetic problem in 2-d or 3-d [6, 8]. By splitting the region of integration at the 'point of observation' integral equation (2) is written in terms of right- and left-going operators L and R with respect to x as

$$\Phi_{inc}(\mathbf{r}_s) = (L + R)\Phi \tag{4}$$

where L and R are defined by

$$Lf(\mathbf{r}) = f - \int_{-\infty}^{\infty}\int_{-\infty}^{x} \frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n} f(\mathbf{r}')\,\gamma(\mathbf{r}')\,dx'\,dy', \tag{5}$$

$$Rf(\mathbf{r}) = -\int_{-\infty}^{\infty}\int_{x}^{\infty} \frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n} f(\mathbf{r}')\,\gamma(\mathbf{r}')\,dx'\,dy' \tag{6}$$

and $\mathbf{r} = (x, y, z)$, $\mathbf{r}' = (x', y', s(x', y'))$. The solution of equation (4) can be expanded as a series,

$$\Phi = (L+R)^{-1}\Phi_{inc} = \left[L^{-1} - L^{-1}RL^{-1} + ...\right]\Phi_{inc}. \tag{7}$$

At moderate to low grazing angles the effect of $\mathbf{R}$ is in some sense small, so that the series converges quickly and can be truncated. This corresponds physically to an assumption that surface-surface interactions are dominated by those 'from the left', as might be expected in this scattering regime. L is large compared with R first, because L includes the dominant 'diagonal' value; second because a predominantly right-going wave gives rise to more rapid phase-variation in the integrand in R than in L. Roughly speaking this occurs because in (2) the phase in the Green's function kernel decreases as the observation point is approached from the left and then increases to the right; whereas the phase of Φ is on average increasing. This is easily verified numerically, and in many cases one or two terms are sufficient to provide an accurate solution.

What is the significance of successive approximations to this field in the ray-theoretic limit? The first iteration contains ray paths which, before leaving the surface, may have interacted with the surface arbitrarily many times but only in a forward direction. The second includes most paths which have changed direction twice: once via the operator R and again via L^{-1}; and so on. The first iteration therefore accounts for multiple scattering, but not *reversible* paths which can occur when incident and backscatter direction are opposite; these paths occur in pairs of equal length and therefore add coherently, giving rise to a peak in the backscattered direction (enhanced backscatter eg [12, 13]) in strongly scattering regimes. Such reversible paths show initially at the second approximation.

Having obtained this series, numerical treatment by surface discretization is straightforward.

NUMERICAL METHOD AND RESULTS

The immediate benefit of series (7) is computational: If the surface is represented using a rectangular grid of M by N points, with M transverse steps (y direction) and N in range (x), then $(L+R)$ becomes an $(MN \times MN)$ matrix, and exact inversion would take $O(M^3N^3)$ operations. By contrast each term of eq. (7) involves inversion of an $M \times M$ matrix at each of N range steps, requiring just $O(NM^3)$ operations and far less memory. Assuming a fixed resolution per wavelength, this scales with λ^4. There is an additional Green function evaluation component, which also increases with λ^4, and in practice this is the dominant computational cost in this approach (typically more than 90% when $M \cong N$).

We briefly sketch the numerical treatment: The notation L, R will be used now for the matrices representing the discretized integral operators as no confusion arises, and we will focus on solution of the first term of (7), i.e. inversion of L. We define discretized surface coordinates x_n, y_m (for convenience using fixed step-sizes Δx, Δy in these directions), giving rise to discretized surface values $\Phi_{inc}(x_n, y_m) = a_{nm}$, $\Phi(x_n, y_m) = b_{nm}$. The resulting matrix L has dimensions $MN \times MN$. This can be viewed as an $N \times N$ lower-triangular block matrix whose entries are $M \times M$ matrices. Gaussian elimination can be used to invert this system by back-substitution. This is an N-step 'marching' process, in which each diagonal $M \times M$ block is inverted in turn; this yields the solution for the unknown surface field by 'marching' in the positive x direction. The details of the algorithm are analogous to those described in [8].

Results: convergence and timings

In order to apply the approach to randomly rough surfaces (for example sea surfaces or terrain), statistically stationary surfaces with Gaussian statistics (normally distributed heights) are generated computationally with a given spatial autocorrelation function (a.c.f.) $\rho(\xi, \eta) =< s(x, y)\ s(x + \xi, y + \eta) >$ where angled brackets denote ensemble averages. In the numerical studies we included the strongly scattering regime of surfaces with correlation length and r.m.s. height both comparable with a wavelength. The tests (apart from the parallel code) were run on a Pentium 4 3.2GHz with 1GB memory, under Linux.

The solution using two terms was compared with full or 'exact' inversion of the boundary integral, and the result is shown in Figure 1. Because of the high computational cost of full inversion this comparison was carried out for a relatively small surface of 12×12 wavelengths, using a grid of 120×120 points. Here the r.m.s. height and correlation length are approximately equal to λ.

The convergence is illustrated by comparison of field values along the mid-line in the x-direction, as can be seen in Figure 2 for the first 4 iterations. Here the field is incident at an angle of 45^o from grazing, and extremely good agreement is produced rapidly. Note that the oscillatory behaviour at the left is captured at the 2nd but not the 1st iteration. It should be noted, however, that convergence cannot be guaranteed for an arbirtrary surface with given statistics; for electromagnetic waves the method [7] exhibited divergences apparently due to resonant surface features.

Computation time: When the algorithm is applied to larger surfaces, the two main contributions to computation time at each of N range steps are are a $N \times N$ matrix inversion and a set of Green's function evaluations. The matrix inversion remains a small percentage of the cost throughout, and overall computation time increases with the square of the number of unknowns, M^2N^2.

Several strategies can be used to reduce Green's function evaluation cost. The most straighforward of these is as follows: A simple quadrature was initially used to carry out integrations using all available surface points. As the spatial separation increases, however, the integrand becomes increasingly smooth as a function of transverse coordinate. Thus the simple quadrature used initially may be replaced by higher order schemes utilizing far fewer

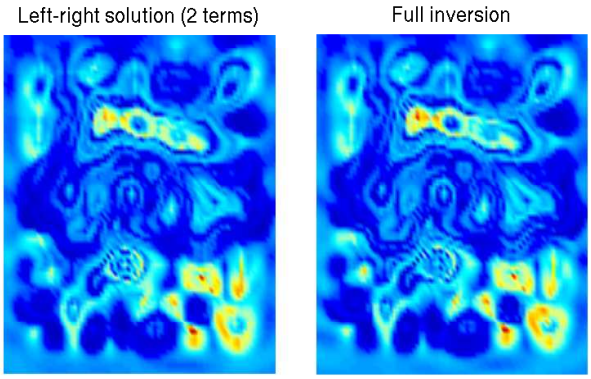

Figure 1: Comparison of surface fields by (a) iterative solution (2 terms), and (b) full inversion for surface with r.m.s. height and correlation length approximately equal to λ.

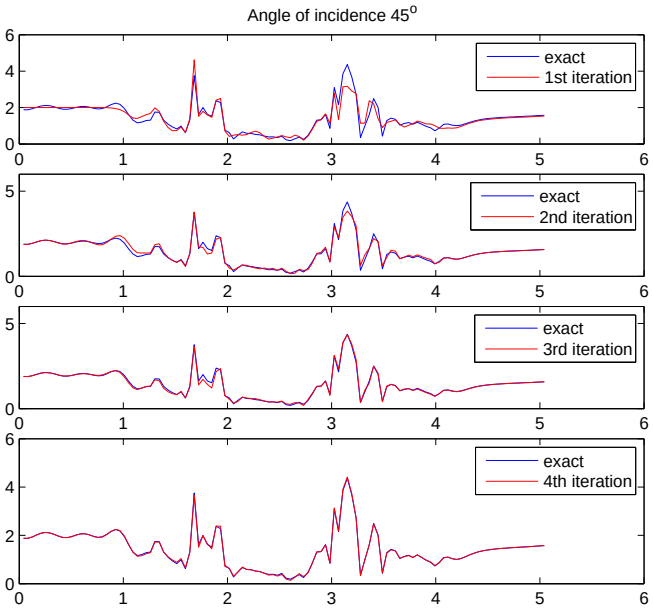

Figure 2: Comparison between exact and successive terms of the left-right solution corresponding to Fig. 1, along a line in x-direction, for grazing angle 45^o.

points and therefore fewer Green's function evaluations, with little loss of accuracy. This does not reduce the *order* of dependence on the number of unknowns, but reduces the multiplier. Even a trapezium rule, for example, operating on half the number of points reduced the computation time by a factor of 3 and resulted in errors of well under 1%. We calculated surface fields on a desktop computer for a surface of $48\lambda \times 48\lambda$ (230,000 unknowns) in around 10 minutes, and $100\lambda \times 100\lambda$ (10^6 unknowns) in 180 minutes. Another possibility is to use fast multilevel multipole. This can reduce the time-dependence to $O(NM \log NM)$, but we found this to have relatively high complexity and memory cost, and accuracy which is not easily regulated.

Parallelisation: The algorithm is easily parallelised, and in particular the quadrature, which takes up most of the computation time, can be shared among any number N_{proc} of processors. This has been carried out using MPI on a Sunfire machine, and as expected the computation speed increases linearly with N_{proc}. Solution for around 5×10^6 unknowns, on a waveguide of 550 λ in length and 80 λ circumference, was obtained in 5.3 hours with standard integration and under 2 hours using optimised integration, on 96 processors.

The method has also been applied to both exterior and interior scattering problems in various geometries, for example by acoustically large waveguides. Most such geometries involve even better-behaved integrals, and are therefore amenable to the above integration optimisation.

SUMMARY

This paper describes the development and application of the left-right splitting algorithm for acoustic scattering by large scatterers. Results have been validated by comparison with "exact" numerical solutions, and by examining the convergence of the series. Although the formulation is especially suited to incident fields at low grazing angles, good convergence has been obtained at angles close to normal incidence. Problems involving up to 10^6 unknowns or more can be solved on a standard desktop computer, and much larger problems in a few hours on a parallel machine. The computational cost scales with the square of the number of unknowns, and although this can be improved by combining the approach with methods such as fast multipole this has not been necessary as in this approach the multiplier is relatively small. The terms in the series represent increasing orders of surface interaction, and can provide insight into multiple scattering mechanisms.

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