ICSV13 - Vienna
The Thirteenth International Congress on Sound and Vibration
Vienna, Austria, July 2-6, 2006

A semi-analytical solution for viscothermal wave propagation in narrow gaps with arbitrary boundary conditions.

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Abstract

Previous research has shown that viscothermal wave propagation in narrow gaps can efficiently be described by means of the low reduced frequency model. For simple geometries and boundary conditions, analytical solutions are available. For example, Beltman [4] gives the acoustic pressure in the gap between an oscillating, rigid, rectangular plate and a rigid surface. Assuming a pressure release boundary condition at the circumference of the plate, excellent agreement with experiments was obtained. In many engineering applications however, the boundary conditions may vary along the circumference of the plate. For instance, the vibrating membranes in hearing aid receivers are attached to complex structures and a simple pressure release ($p = 0$) or zero velocity boundary condition ($dp/dn = 0$) is only valid at some parts of the circumference of the vibrating structure. One can use numerical methods, like FEM or BEM, but often a large number of degrees of freedom is needed to obtain accurate results. Furthermore, a thorough understanding of the various phenomena can only be gained through a large number of calculations. In this paper a semi-analytical solution is presented for the viscothermal wave propagation in the gap between an oscillating, rigid, circular plate and a rigid surface for the arbitrary boundary conditions just mentioned. The pressure in the gap is written as a series expansion of solutions satisfying the differential equations in the interior domain. Subsequently, either the pressure release or the zero velocity boundary condition is imposed on different parts of the circumference. The unknown constants in the series expansion are determined using a weak form of the boundary conditions. It is shown that only a limited number of terms is needed to accurately describe the total acoustic force on the plate. The solution is validated by means of a finite element calculation.

INTRODUCTION

The description of visco-thermal wave propagation in narrow gaps has been described extensively in literature. An overview is given in [2]. A proper description of the phenomenon in the gap is important for understanding the dynamical behavior of, e.g., the stacked solar panels of a satellite during launch or to increase the performance of hearing aid receivers. Beltman showed that from all the models available the low reduced frequency model is the most efficient model. Also the acousto-elastic coupling between flexible plates and a narrow air-layer is now well established [1]. For various simple geometries and simple boundary conditions, one can obtain an analytical solution of the low reduced frequency model. If the geometry of the air layer becomes more complex, a finite element model is the most efficient tool. In [2] it was shown that the agreement between the finite element result, as implemented in the finite element package B2000, and experimental data is good.

However, if the geometry is simple but the boundary conditions are complex, the use of the analytical solutions, as we describe in this paper, may be preferred. Obviously, the analytical solutions give more insight but they also reduce the number of degrees of freedom. This allows the model to be used more efficiently in complex models, where the wave propagation in the air layer is only a sub-problem.

In the finite element method, the differential equations are weighted and the solution only satisfies the differential equation and (some) boundary conditions in a weak form. Our suggested approach is to use the analytical solution, i.e. the differential equation is met analytically in the interior domain, and weight the boundary conditions instead.

The proposed method is used to describe the wave propagation in the gap between an oscillating circular plate and a fixed surface. We will briefly describe the low reduced frequency model and the analytical solution for the pressure in the gap. Next, a Dirichlet boundary condition is imposed on part of the circumference, while on the remaining part a Neumann condition is imposed. The weak form of the boundary conditions is described. The convergence of the solution to the boundary conditions is investigated and solutions are presented for various values of the dimensionless parameters involved. Finally, the method is validated by means of a finite element analysis.

THEORY

Consider a rigid, circular plate of radius R oscillating near a fixed surface, see figure(1). The gap between plate and surface is $\bar{h}(t) = h_0(1+he^{i\omega t})$, where h_0 denotes the mean gap height, h is the dimensionless amplitude of the oscillation, ω is the angular frequency and t denotes time. At the outer circumference of the plate ($\bar{r} = R$) the gap is either open (at $\theta \in \partial\Omega_D$) or closed (at $\theta \in \partial\Omega_N$).

Low reduced frequency model

The low reduced frequency model is used to describe the viscothermal wave propagation in the gap, see e.g. [2]. The model is based on the linearized Navier Stokes equations, the equation of continuity, the equation of state for an ideal gas and the energy equation. It assumes no internal heat generation, a homogeneous medium, laminar flow and only small harmonic perturbations. In addition the acoustic wavelength λ is assumed to be large compared to the

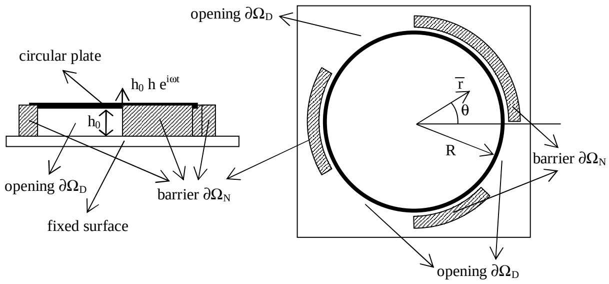


Figure 1: Circular plate oscillating near a fixed surface (front and top view).

mean gap height h_0 and large compared to the viscous boundary layer thickness. For circular coordinates, the low reduced frequency model, in terms of dimensionless variables, reads:

$$iu = -\frac{1}{\gamma}\frac{\partial p}{\partial r} + \frac{1}{s^2}\frac{\partial^2 u}{\partial z^2} \tag{1}$$

$$iv = -\frac{1}{\gamma r}\frac{\partial p}{\partial \theta} + \frac{1}{s^2}\frac{\partial^2 v}{\partial z^2} \tag{2}$$

$$0 = \frac{-1}{k\gamma}\frac{\partial p}{\partial z} \tag{3}$$

$$0 = \frac{k}{r}\frac{\partial (ru)}{\partial r} + \frac{k}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} + ik\rho \tag{4}$$

$$p = \rho + T \tag{5}$$

$$iT = \frac{1}{s^2\sigma^2}\frac{\partial^2 T}{\partial z^2} + i\frac{\gamma - 1}{\gamma}p, \tag{6}$$

where u, v and w denote the velocity perturbation in, respectively, the $r-$, $\theta-$ and z-direction, p is the pressure perturbation, T the temperature perturbation and ρ the density perturbation. The dimensionless variables $(u, v, w, \ldots)$ are related to the physical variables $(\bar{u}, \bar{v}, \bar{w}, \ldots)$ according to: $\bar{u} = uc_0e^{i\omega t}$, $\bar{v} = vc_0e^{i\omega t}$, $\bar{w} = wc_0e^{i\omega t}$, $\bar{p} = p_0(1+pe^{i\omega t})$, $\bar{T} = T_0(1+Te^{i\omega t})$, $\bar{\rho} = \rho_0(1+\rho e^{i\omega t})$, $\bar{z} = zh_0$ and $\bar{r} = rc_0/\omega$, where c_0, p_0, T_0 and ρ_0 denote, respectively, the ambient speed of sound, mean pressure, mean temperature and mean density.

The parameters $\gamma = C_p/C_v$ (C_p is the specific heat at constant pressure, C_v at constant volume) and $\sigma = \sqrt{\mu C_p/\lambda_T}$ (μ denotes viscosity, λ_T thermal conductivity), only depend on the physical properties of the fluid. The parameters determining the solution of the problem are the shear wave number $s = h_0\sqrt{\rho_0\omega/\mu}$, which is a measure for the ratio between the inertial and viscous effects, and the dimensionless radius $\Delta = R\omega/c_0 = 2\pi R/\lambda$, which equals the ratio between the circumference of the plate and the acoustic wavelength $\lambda = 2\pi c_0/\omega$. The low reduced frequency assumption implies that $k = \omega h_0/c_0$, which is a measure for the ratio between the gap height and the acoustic wavelength, is very small.

Based on equations 1 until 6, one can derive the following differential equation for the pressure $p = p(r, \theta)$, see [2]:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} - \Gamma^2 p = 0. \tag{7}$$

where $\Gamma = \sqrt{\gamma/(nB(s))}$ denotes the propagation constant and $n = (1 + B(s\sigma)(\gamma - 1)/\gamma)^{-1}$ is the polytropic constant ($B(\zeta) = (2/(\zeta\sqrt{i}))\,(((cosh(\zeta\sqrt{i}) - 1)/sinh(\zeta\sqrt{i})) - 1)$.

The general solution of equation 7 can be shown to be:

$$p(r, \theta) = C_0 I_0(\Gamma r) + \sum_{m=1}^{\infty}\left(C_{s,m}\sin(m\theta) + C_{c,m}\cos(m\theta)\right) I_m(\Gamma r) - h\;n, \tag{8}$$

were I_m is the modified Bessel function of the first kind of order m and C_0, $C_{s,m}$ and $C_{c,m}$ are constants. Note that the modified Bessel function of the second kind K_m, which is also a solution, is discarded as it becomes infinite at $r = 0$.

Boundary conditions

The pressure in the gap is generally much larger than the pressure outside the gap. Therefore at $\partial\Omega_D$, i.e. where the circumference is open, a pressure release boundary condition, $p(\Delta, \theta) = 0$, is a realistic assumption (Dirichlet). If the barriers are close enough to the oscillating plate, no leakage is possible and the radial velocity perturbation can be set to zero. As the velocity is proportional to the derivative of the pressure, we can set $\partial p(\Delta, \theta)/\partial r = 0$ at $\partial\Omega_N$ (Neumann).

When the boundary conditions are independent of θ, i.e. $\partial\Omega = \partial\Omega_D$ or $\partial\Omega = \partial\Omega_N$, only $C_0 \neq 0$ and the solution is readily obtained. If the boundary $\partial\Omega$ contains both domains $\partial\Omega_D$ and $\partial\Omega_N$, C_0, $C_{s,m}$ and $C_{c,m}$ can not be determined in closed form but can from a weak form of the boundary conditions. For that, the series expansion 8 is truncated to $m = M$ and evaluated at $r = \Delta$ in $\partial\Omega_D$, while in $\partial\Omega_N$ the derivative of the series is evaluated. The results are multiplied by a weight function $w_n(\theta)$ and integrated along $\partial\Omega$. In general, the integral should be set equal to the integral along $\partial\Omega$ of the product of the appropriate boundary conditions and weight function $w_n(\theta)$, which, in the present case, is identical to zero. The constants C_0, $C_{s,m}$ and $C_{c,m}$ can now be determined from a system of equations using a set of weight functions $w_n(\theta)$. In the present study $sin(n\theta)$ and $cos(n\theta)$ have been used for $w_n(\theta)$ and the following system of equations is obtained:

$$\int_{\partial\Omega_D} p(\Delta, \theta)d\theta + \int_{\partial\Omega_N}\frac{\partial p(\Delta, \theta)}{\partial r}d\theta = 0 \tag{9}$$

$$\int_{\partial\Omega_D} sin(n\theta)p(\Delta, \theta)d\theta + \int_{\partial\Omega_N} sin(n\theta)\frac{\partial p(\Delta, \theta)}{\partial r}d\theta = 0 \qquad n = 1 \ldots N \tag{10}$$

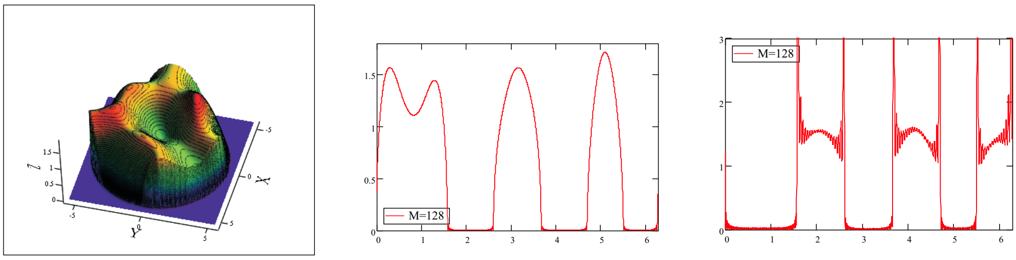
$$\int_{\partial\Omega_D} cos(n\theta)p(\Delta, \theta)d\theta + \int_{\partial\Omega_N} cos(n\theta)\frac{\partial p(\Delta, \theta)}{\partial r}d\theta = 0 \qquad n = 1 \ldots N. \tag{11}$$

where we have chosen $N = M$.

The integrals can be evaluated numerically but straightforward implementation yields high condition numbers for the system matrix when M is large ($I_m(\Gamma\Delta)$ becomes very small for large m). To a large extent, this can be solved by scaling $I_m(\Gamma r)$ by $I_m(\Gamma\Delta)$ and solving for $C_0 I_0(\Gamma\Delta)$, $C_{s,m}I_m(\Gamma\Delta)$ and $C_{c,m}I_m(\Gamma\Delta)$, ensuring that the entries in the system matrix are of the same order.

RESULTS

As an example, we consider a geometry for which $\partial\Omega_N = \theta \in ([0, \pi/2> \cup [5\pi/6, 7\pi/6> \cup [3\pi/2, 7\pi/4>)$ and $\partial\Omega_D = \theta \in ([\pi/2, 5\pi/6> \cup [7\pi/6, 3\pi/2> \cup [7\pi/4, 2\pi>)$. This geometry is shown in figure 1 and chosen because of its lack of symmetry. The dimensionless amplitude of the oscillation of the plate h was set to 1.



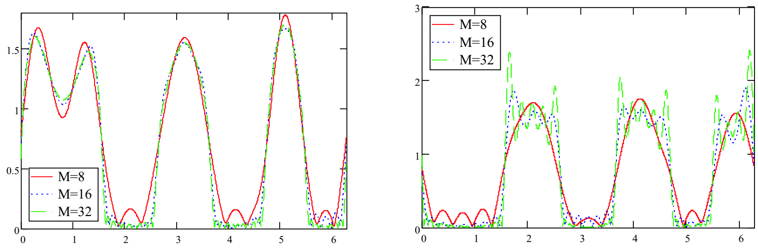
(a) (b) (c)

Figure 2: Pressure amplitude $|p(x, y)|$ (a), $|p(\Delta, \theta)|$ as a function of θ (b) and pressure derivative $|\partial p(\Delta, \theta)/\partial r|$ as a function of θ (c). $s = 5$ and $\Delta = 5$. $M = 128$.

In figure 2, the pressure $|p(x, y)|$, $|p(\Delta, \theta)|$ and the partial derivative of the pressure $|\partial p(\Delta, \theta)/\partial r|$ are shown for $s = 5$ and $\Delta = 5$. As can be seen, the series closely matches the Dirichlet condition $p(\Delta, \theta) = 0$ in $\partial\Omega_D$ and Neumann condition $\partial p(\Delta, \theta)/\partial r = 0$ in $\partial\Omega_N$. Near the points where the boundary domains connect, (i.e. where $\partial\Omega_D$ connects to $\partial\Omega_N$) the pressure rises quickly in Ω_N and the partial derivative $\partial p(\Delta, \theta)/\partial\theta$ becomes singular in $\partial\Omega_D$. Due to the sudden change in boundary conditions, the radial velocity is infinite at that singular point and this results in oscillatory components in the solution. These oscillations can be removed using Lanczos sigma factors [5].

To study the convergence of the solution to the boundary conditions, the pressure and pressure derivative are shown in figure 3 for various values of M. One may conclude that only a limited number of terms is needed to have good agreement with the boundary conditions.

The model described here can be used in a model which also describes the dynamics of the plate, i.e. in a model where h follows from an equation of motion of the plate. Therefore, we are especially interested in the number of terms required to have an accurate representation of the total force acting on the plate. Based on the series expansion 8, the only terms which contribute to the (dimensionless) force F, defined as $F = \int_0^\Delta \int_0^{2\pi} p(r, \theta) r d\theta dr$, are the I_0-



(a) (b)

Figure 3: Pressure amplitude $|p(\Delta, \theta)|$ (a) and pressure derivative $|\partial p/\partial r(\Delta, \theta)|$ as a function of θ (b) for various values of M ($s = 5$ and $\Delta = 5$).

M	1	2	3	4	8
F	-99.8 - 26.9 i	-98.9 - 27.3 i	-90.8 - 26.1i	-90.2 - 25.2 i	-90.3 -24.7 i
M	16	32	64	128	
F	-90.0 - 25.1 i	-89.9 - 25.4 i	-89.8 - 25.7 i	-89.7 - 25.9 i	

Table 1: Dimensionless force F for various values of M.

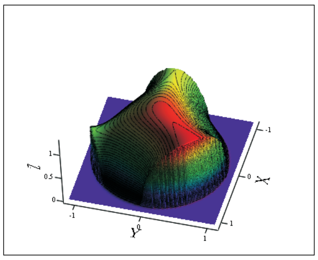
term (C_0) and $h\,n$-term. Note that, unlike a Fourier series, C_0 does vary as the number of terms in the series is increased. Hence the convergence of the total force is determined by the convergence of C_0. In table 1, the total force F is given for various values of M. Surprisingly, one can see that only 7 terms ($M = 3$) suffice to be within 2% of the solution (based upon the solution for $M = 128$). However, the total force converges only slowly as M is further increased. This is attributed to the discontinuity at the points connecting the boundary domains $\partial\Omega_D$ and $\partial\Omega_N$.

Solutions

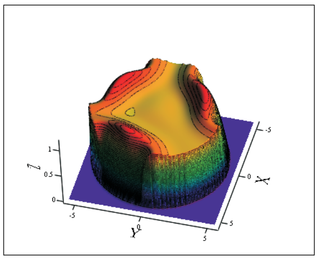
The pressure amplitude $|p(x, y)|$ for various values of the shear wave number s and the dimensionless radius Δ are shown in figure 4.

The effect of the dimensionless radius Δ can be explained as follows. When Δ is small, the circumference (and thus the radius) of the plate is small compared to the acoustic wavelength. Then the pressure distribution is affected by the boundary conditions but remains very smooth. If Δ is increased and becomes comparable to the circumference of the plate, one observes resonant-like behavior, similar to the acoustic resonances observed in enclosures. The exact resonances (frequencies and mode shapes) depend heavily on the distribution of the barriers along the circumference. Note that Δ is a function of the frequency and Δ may thus also be referred to as a reduced frequency (not to be confused with the reduced frequency k). Then certain specific values of Δ can be associated with resonance (eigen-)frequencies. If Δ is further increased the circumference of the plate is much larger than the wavelength and the pressure distribution is highly oscillatory.

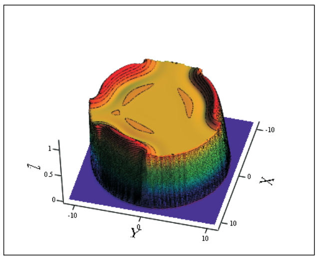
Also the dependency of the the shear wave number s is clear from the figure. When s



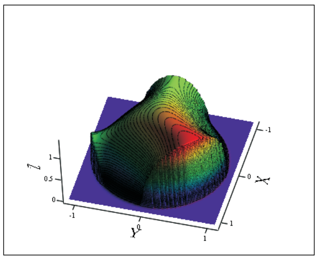
(a) $s = 2, \Delta = 1$



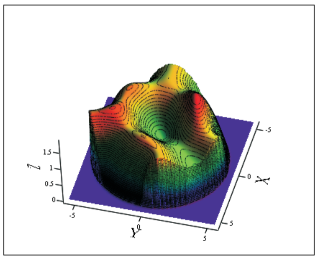
(b) $s = 2, \Delta = 5$



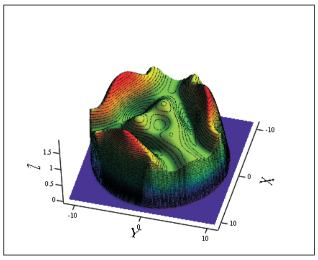
(c) $s = 2, \Delta = 10$



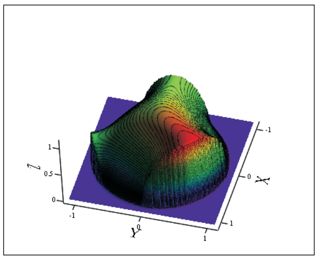
(d) $s = 5, \Delta = 1$



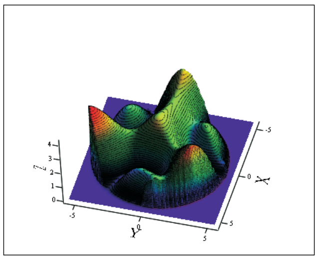
(e) $s = 5, \Delta = 5$



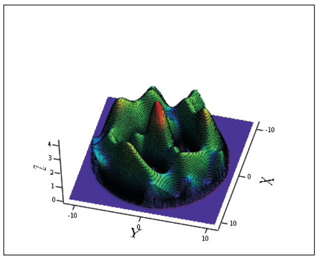
(f) $s = 5, \Delta = 10$



(g) $s = 20, \Delta = 1$



(h) $s = 20, \Delta = 5$



(i) $s = 20, \Delta = 10$

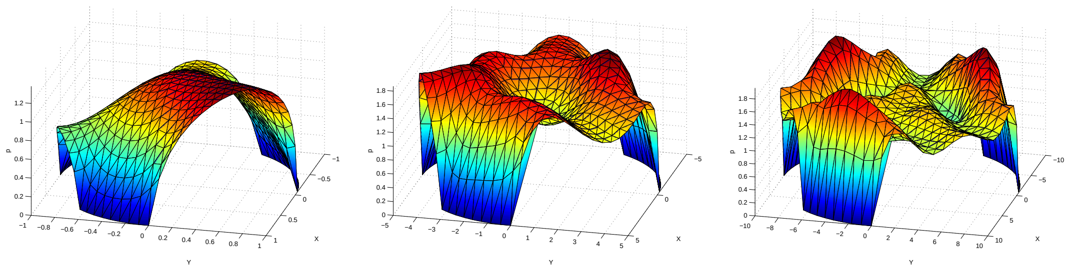
Figure 4: Pressure amplitude $|p(x, y)|$ for various values of the shear wave number s and dimensionless radius Δ.

is small, the thermal- and viscous effects are large. Then the plate squeezes fluid/air in and out of the openings, similar to a viscous pump, and the associated pressure is shown in figure 4(i). In this case, the boundary conditions affect the solution only close to the boundary. For high values of s the fluid behaves non-viscous and boundary conditions affect the solution in the entire domain.

Validation using FEM

The results shown in the previous sections have been validated using a finite element implementation of the low reduced frequency model for the given geometry. Figure 5 shows the absolute value of the pressure $|p(x, y)|$ for $s = 5$ and various values of Δ. The results have been obtained using 436 degrees of freedom.

If figure 5 is compared to figure 4, one can conclude that the semi-analytical method,



(a) (b) (c)

Figure 5: FEM results for the pressure $|p(x, y)|$ for $\Delta = 1$ (a), 5 (b) and 10 (c).

as proposed, is validated. In addition, one may conclude that the semi-analytical solution provides more detail with less degrees of freedom.

CONCLUSIONS

In this paper, a semi-analytical approach for the description of wave propagation in narrow gaps with arbitrary boundary conditions is presented. The method uses the analytical solutions for the pressure in the gap (in terms of a series expansion). The constants in the series expansion are determined by imposing boundary conditions in a weak form. This method can be used when the geometry of the gap is simple but arbitrary boundary conditions apply. Then, it can be used as a fast alternative compared to a finite element approach.

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