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PHYSICAL MECHANISM BEHIND THE EFFECT OF VANISHINGLY SMALL DAMPING ON THE STABILITY BOUND OF BECK'S COLUMN

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Abstract
The paper gives a physical explanation of the mechanism behind the so-called destabilizing effect of small internal damping in the dynamic stability of Beck's column. Both internal (material) and external (viscous fluid) damping is considered. An energy equation is derived for the balance between the work done by the nonconservative 'follower force' and the energy dissipated by the internal and external damping forces. Evaluated at the critical load, where a flutter instability is initiated, this equation explicitly shows the influence of damping upon flutter frequency, phase angle, and vibration amplitude. The gradient of the phase angle, evaluated at the free end of the column, is found to be the 'valve' which controls how much work the follower force can do on the column during each period of oscillation. And a large change in this gradient with increasing - but still small - internal damping is found to be responsible for the destabilizing effect.

INTRODUCTION

Ziegler [8] discovered in 1953 that damping may act destabilizing in nonconservative systems. He considered a double pendulum subjected to a tangential 'follower force' at the free end and found that the critical load, where a dynamic instability (flutter) is initiated, is higher for zero damping than for vanishingly small but non-zero damping. This appears to oppose 'common sense' and caused thus a great deal of interest, which has continued till date. An overview of the many published studies can be found in the recent review papers [4, 2]. The emphasis has mostly been, and is still, on a mathematical understanding of the eigenvalue behavior near the singularity at zero damping [3]. But, despite the amount of work done on this topic, the physical mechanism behind the destabilizing effect is not yet well understood. It is desirable to know how this mechanism is manifested in physically (experimentally) observable quantities, such as frequency, amplitude, and phase.

Semler *et al.* [6] have recently given an interesting physical explanation of the destabilization effect in Ziegler's double pendulum. Their analysis and physical arguments are based on the energy balance between energy input from the follower force and dissipation by damping, and illustrates the significance of phase and amplitude relations.

The present paper gives an physical explanation of the destabilization effect in a continuous system by considering Beck's column, as shown in Fig. 1, using a continuous (non-discretized) energy analysis, as opposed to the modal-based analysis of ref. [6]. Beck's column is more practical and realistic than Ziegler's pendulum model, as it can be considered as a simplified model of a rocket-propelled flying beam [5]; and it can be realized experimentally to good approximation through a cantilevered column with a small, powerful solid-propellant rocket motor mounted at the free end [7].

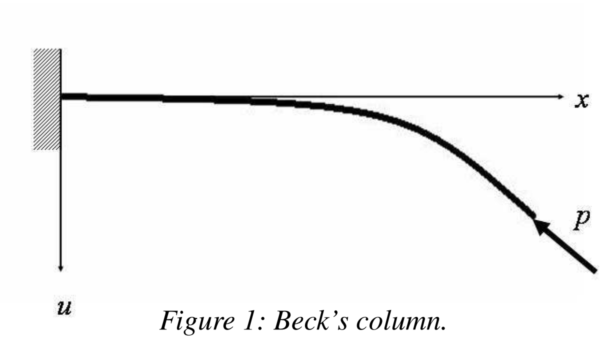

Figure 1: Beck's column.

EQUATION OF MOTION AND BOUNDARY CONDITIONS

The equation of motion for small-amplitude vibrations of a uniform Beck's column with internal (Kelvin-Voigt) and external (viscous) damping is, in non-dimensional form, given by

$$\frac{\partial^2 u}{\partial t^2} + \beta \frac{\partial u}{\partial t} + \gamma \frac{\partial^5 u}{\partial t \partial x^4} + \frac{\partial^4 u}{\partial x^4} + p \frac{\partial^2 u}{\partial x^2} = 0. \tag{1}$$

Here $u(x,t)$ is the deflection at position x and time t, p is the load parameter, while β and γ are external and, respectively, internal and damping parameters. Assuming that the column is clamped at $x = 0$ and free at $x = 1$, the four boundary conditions are

$$[u]_{x=0} = 0, \quad \left[\frac{\partial u}{\partial x}\right]_{x=0} = 0, \tag{2}$$
$$\left[\gamma \frac{\partial^3 u}{\partial t \partial x^2} + \frac{\partial^2 u}{\partial x^2}\right]_{x=1} = 0, \quad \left[\gamma \frac{\partial^4 u}{\partial t \partial x^3} + \frac{\partial^3 u}{\partial x^3}\right]_{x=1} = 0.$$

A time dependence on the form $u(x,t) = \hat{u}(x)\exp(\lambda\, t)$, $\lambda = \alpha + \mathrm{i}\,\omega$ is assumed. The eigenvalue pair (λ, p) can be determined numerically without discretizing the problem. However, for ease of analysis, the problem is discretized using the Galerkin finite element method. The critical load (or flutter load), denoted by p_{cr}, is defined as the smallest value of p for which the real part of an eigenvalue $\alpha = \mathrm{Re}(\lambda)$ crosses the $\alpha = 0$ line, such that $\alpha > 0$ for $p > p_{cr}$. The critical load is determined numerically by bisection. The real part α is seen to represent an amplitude growth factor. The to $\alpha = 0$ corresponding imaginary part $\omega = \mathrm{Im}(\lambda)$ is termed the flutter frequency, and is denoted by ω_{cr}. Divergence ($\alpha > 0$, $\omega_{cr} = 0$) is not possible by Beck's column.

STABILITY DIAGRAMS

Fig. 2 reproduces and reviews some well-known but basic results, which are essential for the further discussions. Part (a_1) shows the critical (flutter) load p_{cr} and part (a_2) the corresponding flutter frequency ω_{cr} as function of the internal damping parameter γ ($10^{-6} \leq \gamma \leq 0.1$), with no external damping ($\beta = 0$). It is seen that both p_{cr} and ω_{cr} remain practically constant by small γ-values - over four decades ($10^{-6} \leq \gamma \lessapprox 0.01$). In this range, $p_{cr} \approx 10.94$ and $\omega_{cr} \approx 5.40$. It is noted that the critical load for the completely undamped column $p_{cr} \approx 20.05$, with corresponding flutter frequency $\omega_{cr} \approx 11.02$. The jump in p_{cr} from 20.05 to 10.94 when vanishingly small (but non-zero) internal damping is introduced is known as the 'destabilization paradox' of small damping.

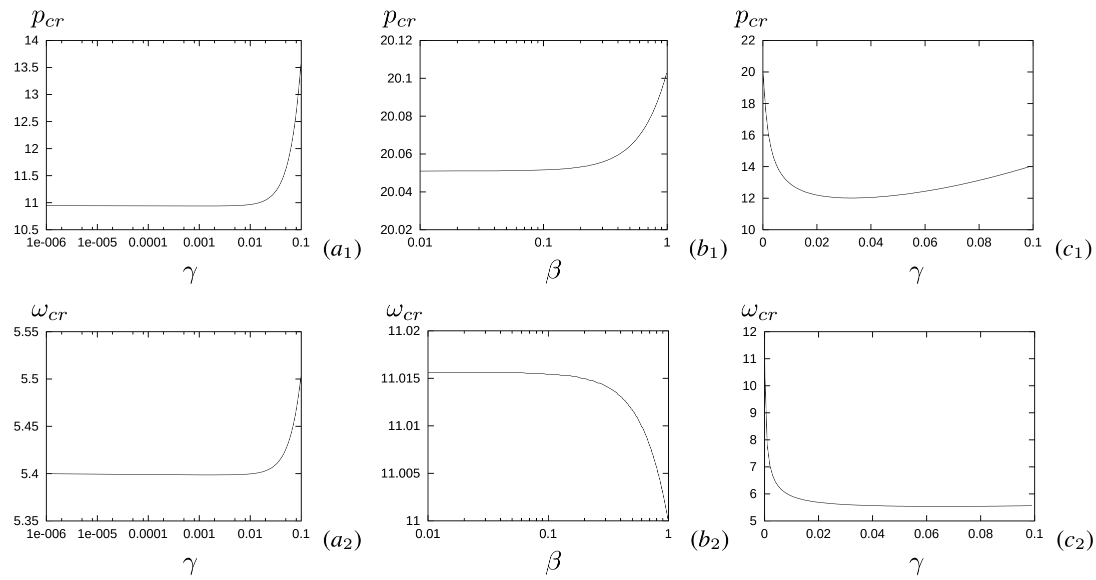

Figure 2: Critical (flutter) load p_{cr} and corresponding flutter frequency ω_{cr}. (a): As function of γ ($\beta = 0$); (b): as function of β ($\gamma = 0$); (c): as function of γ ($\beta = 0.1$).

Corresponding results for external, viscous damping only are given in parts ($b_{1,2}$). It is seen that this type of damping alone has a stabilizing effect. [This can also be proved mathematically.]

The dependence of the critical load p_{cr} on the internal damping factor γ becomes more complicated when external damping also is included. This is illustrated by parts ($c_{1,2}$), which show p_{cr} and ω_{cr}, respectively, as functions of internal damping γ with the external damping parameter β =0.1. Increasing the amount of internal damping has a destabilizing effect for small values of γ and a stabilizing effect for larger values. At what value of γ the change in behavior occurs depends on the value of β, that is, the mutual balance between β and γ is important.

ENERGY CONSIDERATIONS

Multiplication of (1) by the lateral velocity of the column $\partial u/\partial t$, followed by integration over the length ($0 \leq x \leq 1$) gives a power (rate of work) balance equation. When the boundary conditions (2) are taken into account via integration by parts, this equation can be written as

$$\frac{d}{dt}\left(T + V - W_c\right) = \frac{dW_{nc}}{dt} + \frac{dW_d}{dt}. \tag{3}$$

The left hand side represents the rate of increase of mechanical energy E (E = kinetic energy T + potential energy $V - W_c$, where V is the elastic energy W_c the work done by the conservative part of the follower force). The individual terms are not given here, due to the space limitation. The right hand side represents the source responsible for this energy increase. The power delivered by the nonconservative component of the follower force is

$$\frac{dW_{nc}}{dt} = -p\left[\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right]_{x=1} \tag{4}$$

and the power 'delivered' (i. e., *minus* the power dissipated) by the damping forces is

$$\frac{dW_d}{dt} = -\int_0^1 \left[\beta\left(\frac{\partial u}{\partial t}\right)^2 + \gamma\left(\frac{\partial^3 u}{\partial t \partial x^2}\right)^2\right] dx. \tag{5}$$

Following Benjamin [1], motion of the column over a time interval $[t_1, t_2]$ will be considered, where the shape of the column at time t_2 coincides with the shape at time t_1. The increase in mechanical energy is then given by

$$\Delta E = \underbrace{-\int_{t_1}^{t_2} p\left[\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right]_{x=1} dt}_{\Delta W_{nc}} \quad \underbrace{-\int_{t_1}^{t_2}\int_0^1 \left[\beta\left(\frac{\partial u}{\partial t}\right)^2 + \gamma\left(\frac{\partial^3 u}{\partial t \partial x^2}\right)^2\right] dxdt}_{\Delta W_d}. \tag{6}$$

Exactly at the critical load, where $\alpha = 0$ and $\omega = \omega_{cr}$, harmonic vibrations exist, such that $\Delta E = 0$ in (6). These vibrations can be expressed as

$$u(x,t) = \mathrm{Re}\left[\hat{u}(x)\exp(\mathrm{i}\omega_{cr}t)\right] = A(x)\cos(\omega_{cr}t + \phi(x)), \tag{7}$$

where $A(x)$ is the amplitude function and $\phi(x)$ the phase angle function. Inserting (7) into

(6), and carrying out the time integration with $t_1 = 0$, $t_2 = 2\pi/\omega_{cr}$, gives

$$\Delta E = 0 = \underbrace{-p_{cr}\pi A(1)^2 \left[\frac{d\phi}{dx}\right]_{x=1}}_{\Delta W_{nc}} \quad \underbrace{-\omega_{cr}\beta\pi \int_0^1 A(x)^2 dx}_{\Delta W_{de}} \tag{8}$$

$$\underbrace{-\omega_{cr}\gamma\pi \int_0^1 \left[\left\{ 2\frac{dA}{dx}\frac{d\phi}{dx} + A\frac{d^2\phi}{dx^2} \right\}^2 + \left\{ \frac{d^2A}{dx^2} - A\left(\frac{d\phi}{dx}\right)^2 \right\}^2 \right] dx}_{\Delta W_{di}} \,.$$

In this equation, the work done by the damping forces ΔW_d has been separated into an external and an internal part, denoted by ΔW_{de} and ΔW_{di}, respectively. These are negative definite, and the phase angle gradient $[d\phi/dx]_{x=1} < 0$ in order to satisfy (8).

Flutter configurations and corresponding phase angle functions are shown in Fig. 3 for three different damping cases. Part (*a*) is for the undamped case. The flutter configurations are seen to be a linear combination of, primarily, first and second eigenmode. Here $[d\phi/dx]_{x=1} = 0$, implying that the follower force does no net work on the column during one period of oscillation. A $180°$ phase shift is seen to occur at the nodal point. In part (*b*) a non-large amount of external damping ($\beta = 0.1$) is included. The flutter configurations keep their basic (undamped) character, but the phase angle distribution is somewhat 'smoothed out', and a small negative gradient ($[d\phi/dx] < 0$) along the column can be recognized. A small amount of internal damping ($\gamma = 2 \times 10^{-4}$) has been added in part (*c*). This is seen to suppress the influence of the second eigenmode, and further 'smear out' the phase angle distribution.

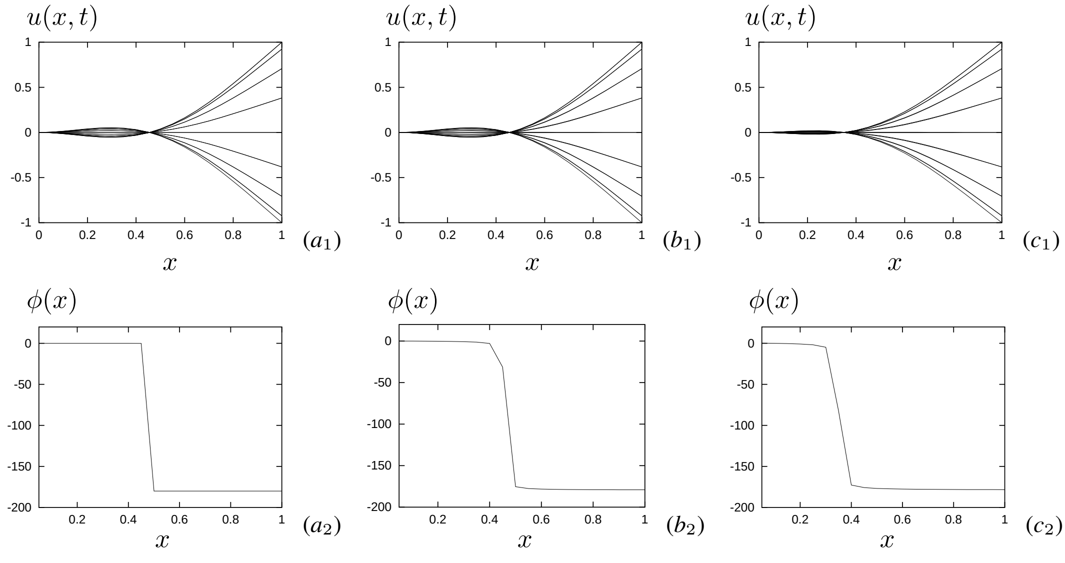

Figure 3: Flutter configurations (a$_1$-c$_1$) and corresponding phase angle functions (a$_2$-c$_2$). (a): $\gamma = \beta = 0$; (b): $\beta = 0.1$, $\gamma = 0.0$; (c): $\beta = 0.1$, $\gamma = 2 \times 10^{-4}$.

It is remarked that the *wave number* $k(x)$ of a wave travelling along a beam is defined as minus the gradient of the phase angle, that is, $k(x) = -d\phi/dx$. The work done by the follower force varies thus as $\Delta W_{nc} \sim p_{cr}\,k(1)$. This shows that introduction of damping in Beck's column introduces a *travelling wave* (progressive wave).

Fig. 4(a) shows how the phase angle distribution $\phi(x)$ along the column develops with increasing load when a (vanishingly) small amount of internal damping is included ($\gamma = 10^{-6}$). It is seen that an increase of the follower force increases the variation in $\phi(x)$.

It has been found numerically that the relation between the gradient and the internal damping parameter γ is approximately linear, provided γ is sufficiently small. That is to say, when only internal damping is present, $[d\phi/dx]_{x=1} \approx -c_1\gamma$, where c_1 is a positive constant. [The graph is not shown due to the space limitation.] Inserting this expression into the first term of (8) gives $\Delta W_{nc} \sim p_{cr}c_1\gamma$, that is, the work done by the follower force is directly proportional to the damping coefficient. Eq. (8) then gives (for $\beta = 0$) that *the magnitude of the flutter load is independent of the actual value of the damping coefficient*. This proves mathematically what is indicated numerically in Fig. 2(a).

When external damping (β) is included the gradient function $[d\phi/dx]_{x=1}$ is still approximately a linear function of γ, as shown in Fig. 4(b), which is for $\beta = 0.1$. The destabilizing effect of small internal damping illustrated in Fig. 2(c) will be considered in the following, making use of this fact, and the work balance shown in Fig. 4(c).

Introduction and increase of internal damping implies naturally an increasing value of its work $-\Delta W_{id}$. In the case considered in Fig. 4(c), the increase of γ implies however that the work done by the external damping $-\Delta W_{de}$ decreases, and at a larger rate than $-\Delta W_{id}$ increases. [The drop in $-\Delta W_{de}$ is mainly due to the reduction of the flutter frequency (Fig. 2) and the reduction of the second eigenmode-component (Fig. 3).] As a consequence, the work done by the follower force ΔW_{nc} must decrease to keep the balance. As $[d\phi/dx]_{x=1}$ decreases, the critical load p_{cr} will necessarily have to decrease also. By further increase in γ the drop in $-\Delta W_{de}$ starts to 'flatten out', and for $\gamma > 0.001$, ΔW_{nc} actually starts to increase. But the steep gradient of $[d\phi/dx]_{x=1}$ implies that p_{cr} necessarily must continue to decrease. In this way the destabilizing effect of internal damping continues until $\gamma \approx 0.03$ (Fig. 2c).

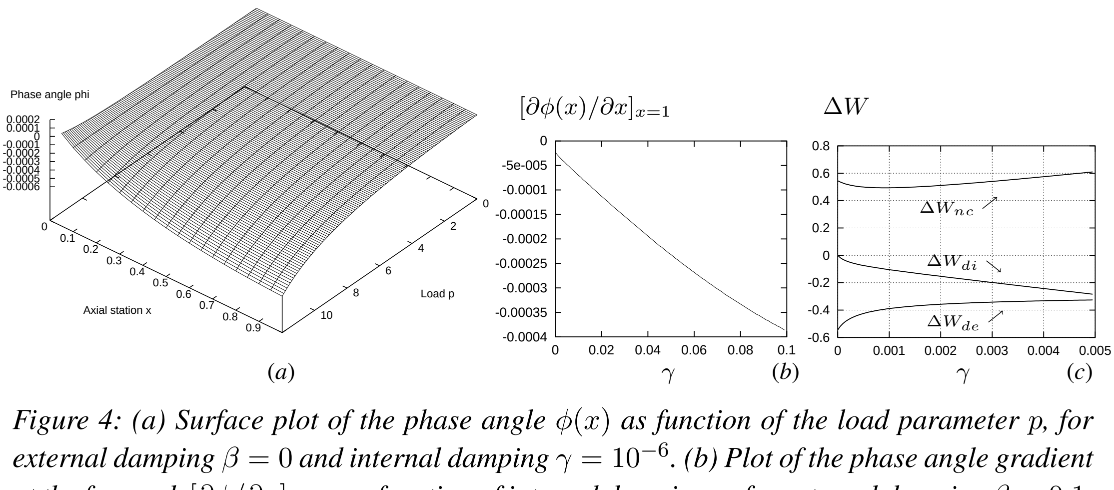

Figure 4: (a) Surface plot of the phase angle $\phi(x)$ as function of the load parameter p, for external damping $\beta = 0$ and internal damping $\gamma = 10^{-6}$. (b) Plot of the phase angle gradient at the free end, $[\partial\phi/\partial x]_{x=1}$, as function of internal damping γ, for external damping $\beta = 0.1$. (c) Work balance as function of internal damping γ, for external damping $\beta = 0.1$.

Introduction of internal damping at the undamped flutter load level

The final part of the paper considers the destabilizing effect by introduction of vanishingly small but non-zero internal damping to the (initially) undamped column. As previously noted, this implies a jump (discontinuous change) in the critical load, from $p_{cr} = 20.05$ to 10.94. We can avoid this jump by taking the undamped column as starting point and gradually increase the internal damping γ, keeping the load parameter p on the undamped critical value $p_* = 20.05$. Fig. 5(a) shows how the real part, $\alpha_* = \mathrm{Re}(\lambda_*)$, of the leading (most unstable) eigenvalue λ_* increases rapidly as the internal damping parameter γ is increased, implying instability. Fig. 5(b) shows the corresponding flutter frequency, which is seen to decrease. This is because internal damping suppresses the second (and higher) eigenmode components of the flutter configurations.

A physical explanation of the increasing α_*-value is provided by Fig. 5(c), which shows the graphs of the energy input from the follower force ΔW_{nc} and the dissipation by internal damping ΔW_{di}. It is seen that ΔW_{nc} increases much more rapidly than $-\Delta W_{di}$; hence the instability.

The reason for the increase in ΔW_{nc} is the decreasing value of $[d\phi/dx]_{x=1}$, as shown in Fig. 5(c). It could be suspected that $-\Delta W_{di}$ is not able to keep up with ΔW_{nc} because the flutter frequency ω_* is decreasing. It has however been found that $-\Delta W_{di}$ is not able to keep up with ΔW_{nc} even if the flutter frequency for $\gamma = 0$ is used consequently. The drop in ω_* has thus little significance in the rapid increase of α_*. The main physical reason for the instability, and hence for the destabilizing effect of small internal damping, is the significant drop in $[d\phi/dx]_{x=1}$ (Fig. 5d); this opens up for the 'ΔW_{nc}-valve' much more efficiently than than the dissipation ΔW_{di} is able to keep up with.

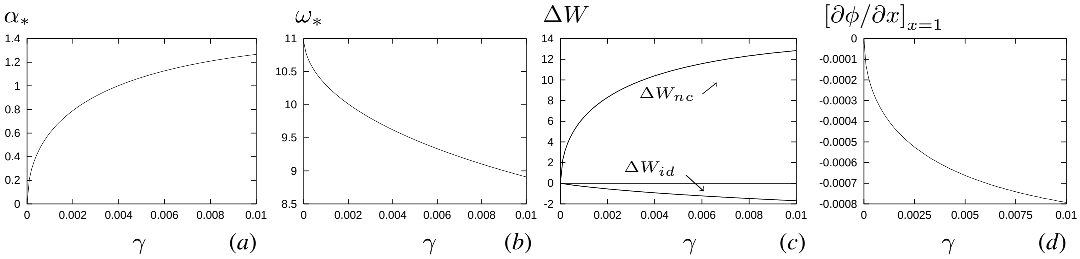

Figure 5: (a) Real part $\alpha_* = \mathrm{Re}(\lambda_*)$. (b) Imaginary part $\omega_* = \mathrm{Im}(\lambda_*)$. (c) Energy input ΔW_{nc} and dissipation ΔW_{di}. (d) Phase angle gradient at the free end, $[\partial\phi(x)/\partial x]_{x=1}$.

CONCLUDING REMARKS

Damping has the dual role of providing a mechanism for energy dissipation ('stabilization') and a mechanism for energy input from the follower force ('destabilization'), and it is in this sense that damping may act destabilizing. The present paper has attempted to clarify these aspects through an energy analysis. The main findings can be summarized as follows.

1. The work done by the follower force at the flutter bound, ΔW_{nc}, is proportional to

(minus) the gradient of the phase angle function at the free end, $-[\partial\phi/\partial x]_{x=1}$, which equals the the wave number there, $k(1)$. Introduction of damping (internal and/or external) in Beck's column introduces thus a travelling wave component in the flutter configurations. The follower force can only do work on the column during each cycle of oscillation if the column vibrates with a travelling wave component.

2. The gradient function $[\partial\phi/\partial x]_{x=1}$ decreases (approxomately) linearly with increasing internal damping γ, provided that γ is sufficiently small. Using this fact it is shown mathematically that the critical load is independent of the actual value of γ, again provided that γ is sufficiently small. The numerical results agree with this.

3. The destabilizing effect of small internal damping has been shown to be caused by the rapid decrease of the phase angle gradient at the free end, $[\partial\phi/\partial x]_{x=1}$, which implies that the critical follower load p_{cr} must decrease in order to maintain the energy balance with the damping forces, as the dissipation only increases moderately with γ. [In the example of Fig. 4, with external damping included, it has even been found that an increase in external damping γ may lower the total energy dissipation $\Delta W_d = \Delta W_{di} + \Delta W_{de}$.] An 'alternative' demonstration has been given by gradually increasing γ while keeping the load level on the critical value for the undamped column, $p_* = 20.05$. Here it has been found that ΔW_{nc} increases much more rapidly than $-\Delta W_{di}$, again because of the rapid decrease in $[\partial\phi/\partial x]_{x=1}$.

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