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MODELLING OF OSCILLATORY SYSTEMS WITH VISCOUS AND DRY-FRICTION DAMPING UNDER REAL RANDOM KINEMATICAL EXCITATION

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Abstract

The paper deals with kinematically excited single DOF oscillatory system, incorporating both viscous and dry-friction vibration damping. By simulation their influence on system properties will be analysed. Two approaches will be treated – one using the signum function approximation, the second one using switching between slipping and sticking states. Both models, exposed to random excitation, will be compared to a SDOF system without friction.

INTRODUCTION

Dry friction is omnipresent in mechanical systems and concern of scientific research for centuries [1, 4]. The first scientific work on this issue is attributed to Coulomb in 1785, however Leonardo da Vinci's genius tackled this problem long before [4]. Despite of long standing research the mathematical description of this phenomenon is not fully furnished and its simulation description is by no means simple. The phenomenon is not always reproducible, as its extent depends on surfaces state, surfaces lubrication, asperities, temperature, normal force magnitude, relative velocity, etc. [1, 2, 6, 10]. The first rigorous analytical treatment of this problem is attributed to Den Hartog, back in 1931 [3], whose work is still much coveted [5]. Various approaches to this problem are presented in the literature, e.g. [2, 9-13]:

- static friction models [9, 11, 13],
- dynamic friction models whose cater for time-delayed friction force and its hysteresis, as well as all the details of the surfaces contact mechanics. Some hitherto used models are the Dahl model; the LuGre model; the Leuven model; the Petrov - Ewins model [10]. All these dynamic models assume detailed knowledge of sliding surfaces parameters. They are mentioned for sake of completeness only.

Here the first approach will be followed. Basic mathematical description of Coulomb friction is by the relay function, or signum function:

$$F_{\mathrm{f}} = F_{\mathrm{fk}} \operatorname{sign}(v_{\mathrm{r}}), \tag{1}$$

where F_{f} is the resulting friction force with varying direction, F_{fk} is the Coulomb friction force and v_{r} is the relative velocity of the sliding surfaces.

A/ Coulomb model involves a proportional dependence of the Coulomb friction force F_{fk} on the normal loading force F_{N} [2, 9, 14], which is assumed to be constant:

$$F_{\mathrm{fk}} = \mu_{\mathrm{k}} F_{\mathrm{N}}, \tag{2}$$

where μ_{k} is the kinetic friction coefficient given for various surfaces in the literature.

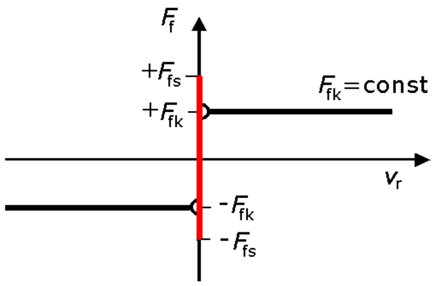


Figure 1 – Friction force F_f courses as function of the relative velocity v_{r}

The friction force at zero relative velocity cannot be determined, i.e. the friction force F_{f} for $v_{\mathrm{r}} = 0$ can have any value in the interval $(-F_{\mathrm{fk}}, +F_{\mathrm{fk}})$. Signum function is often [2, 6, 8, 11, 14] mathematically described as:

$$\operatorname{sign}(v_{\mathrm{r}}) = \left\langle \begin{matrix} +1 & \text{for } v_{\mathrm{r}} > 0 \\ -1 & \text{for } v_{\mathrm{r}} < 0 \end{matrix} \right. . \tag{3}$$

However, different authors define differently function value for argument value $v_{\mathrm{r}} = 0$. Note also that sign function, as defined by expression (3) has no limit for $v_{\mathrm{r}} = 0$ and is neither differentiable for $v_{\mathrm{r}} = 0$, hence is not a “neat” continuous function.

B/ In reality a larger force $F_{\mathrm{fs}} > F_{\mathrm{fk}}$ is needed for overcoming the adhesion at zero relative velocity to start the sliding motion [2]. The friction force at $v_{\mathrm{r}} = 0$ has to be described as a function of a limit force F_{L}, external to the dry-friction interface. The limit force F_{L} is obtained by analysing the force balance across the sliding surfaces interface and has to be compared to the static friction force value F_{fs}:

$$\text{if } |F_{\mathrm{L}}| \le F_{\mathrm{fs}} \Rightarrow v_{\mathrm{r}} = 0. \tag{4}$$

If this condition is met the system is in standstill in the so-called stick state, indicated in Figure 1 by the red line segment. If at a certain time instant the adhesion force F_{fs} is overcomed by the external force F_L the oscillatory systems starts abruptly to move and the relative velocity v_r attains some non-zero value. Further on the Eq. (1) is valid until v_r eventually decreases to zero and the system stops again. This start-sliding-stop movement (stick-slip movement) leads to non-unique solution of equations describing the motion and pose mathematical difficulties [6-8, 11, 14]. Analogically, the static friction coefficient μ_s is defined as $\mu_s = F_{fs}/F_N$; while $\mu_s > \mu_k$.

C/ Stroke (relative displacement amplitude) x_{ra} limitations of any technical oscillator due to design constraints to a maximal value x_{rM} have to be accounted for:

$$x_{ra} < x_{rM} . \tag{5}$$

If at any time instant $x_{ra}(t) \geq x_{rM}$ the structure is hard hit and impact phenomena with high acceleration peaks would occur, leading to possible chaotic behaviour. In practical systems measures are taken to avoid this situation and soften the end-stop impact, as described e.g. in [16]. In this paper the fulfilment of the constraint will be just checked without going into analysis of the end-stops influence.

MODEL DESCRIPTION

A horizontal oscillatory system, whose schematic layout is depicted in Figure 2, will be further analysed. The system will be subjected to kinematic excitation by displacement $u(t)$ and its respective time derivatives. This is a so-called mixed-mode dissipative system [3], as both a linear viscous damper with damping coefficient b is present and the dry-friction force F_f is acting at the sliding surfaces interfaces.

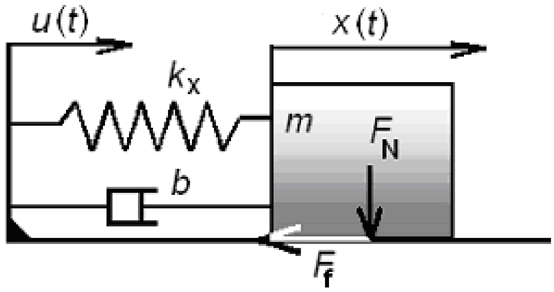


Figure 2 – Schematic diagram of the analysed oscillatory system

The equation of motion in the sliding mode is ($x_r = x - u$ is the relative displacement):

$$m\ddot{x} + k_x x_r + F_d = m\ddot{x} + k_x x_r + \left[b\dot{x}_r + F_{fk}\mathrm{sign}(\dot{x}_r)\right] = 0 , \tag{6}$$

where: F_d represents the mixed mode damping force,

$\dot{x}_r \equiv v_r$ is the relative velocity across sliding surfaces.

The exerted normal force F_N is usually constant and equal to mass m weight, but F_N may be time dependent, too.

Following approaches to this system analysis and simulation can be used:

1. Using signum function (Eq. (3)) or its continuous approximation [9, 10], while neglecting the $v_r = 0$ state.
2. Performing the force-balance analysis at $v_r = 0$ and solving the respective describing equation in each system state (sticking or sliding) [11, 13].

1. In using the signum function after Eq. (3) the modified Eq. (6) is used, assuming non-zero relative velocity v_r:

$$m\ddot{x}_r + k_x x_r + \left[b\dot{x}_r + F_{fk}\mathrm{sign}(\dot{x}_r)\right] = -m\ddot{u}\,. \tag{7}$$

For $v_r = 0$ the sign function is set to zero. Analysis for $v_r = 0$ is completely neglected. A "smoothed" continuous function is sometimes used instead [9, 10]:

$$\mathrm{sign}(v_r) \approx \tfrac{2}{\pi}\arctan(c\cdot v_r) \approx \tanh(c\cdot v_r) \approx \mathrm{erf}(c\cdot v_r) \approx \frac{c\cdot v_r}{1+c\cdot|v_r|}\,. \tag{8}$$

Constant c in each of the functions describes the numerical "match" between the sign function and the continuous function used for approximation. In [9] it is demonstrated, that a value of $c \geq 10^3$ suffices and the difference to analytical solution [3] is less than 1 %. It is suggested that the last formula is better to attain good computational speed at the same level of accuracy. This approach circumvents the problem of solving differential Eq. (7) with the discontinuous signum function.

2. The slip-stick transients have to be accounted, if physically correct description of the friction process is sought. This is furnished in following way:

i. For $v_r \neq 0$ Eq. (7) is valid.

ii. When the $v_r \neq 0$ to $v_r = 0$ transition occurs the movement stops and the force balance condition across the friction interface, described by Eq. (4), has to be tested, while:

$$F_L = m\ddot{u} + k_x x_r\,. \tag{9}$$

Until $|F_L| \leq F_{fs} \Rightarrow v_r = 0$ and no relative movement occurs.

As soon as $|F_L| > F_{fs} \Rightarrow v_r \neq 0$, the dry friction force abruptly decreases to F_{fk} and Eq. (7) has to be re-solved with actual initial conditions.

Sometimes the distinction between the slip and stick states is facilitated by a set of more precise conditions [11, 13]:

Slipping: $|v_r| > \varepsilon \quad |F_L| > F_{fs}$, (10a)

Sticking: $|v_r| < \varepsilon$ AND $|F_L| < F_{fs}$. (10b)

The operation for evaluation of conditions (10) is called “variable zero-crossing” and is facilitated in the MATLAB/Simulink® by specific procedures [15]. The main difficulty for equidistantly paced numerical systems, in contrary to analytical considerations, is the need of precise determination of the time instant, when the zero crossing occurs, or when $|v_r| < \varepsilon$, while the value of ε has to be assessed independently. This approach is very important when equidistantly sampled experimental data have to be processed. The MATLAB® standard stiff ordinary differential equations solvers with variable time increment are not applicable. Hence another approach was followed and a proprietary fixed equidistant step ordinary differential equations solver has been developed, which specifically caters for the determination of the $|v_r| < \varepsilon$ condition within the given fixed time increment Δt.

EXAMPLE OF APPLICATION OF THE ABOVE APPROACHES

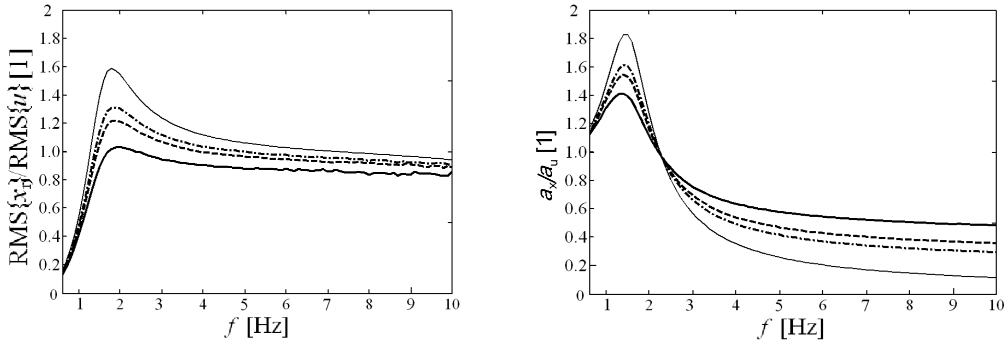
An example of the described simulation approaches will be given, using Eq. (7), with following parameters: mass $m = 75$ kg, spring stiffness $k_x = 7500$ N/m, giving undamped natural angular frequency of $\omega_0 = 10.0$ rad/s, i.e. natural frequency of $f_0 = 1.592$ Hz. Let assume a linear damping coefficient of a value of $b = 500$ Ns/m (i.e. damping ratio $\xi = 0.333$) and a constant normal force F_N, equal to the body weight of 735.75 N. The maximal relative displacement is limited to 25 mm. Two cases will be analysed, denoted as “low dry-friction” and “high dry-friction” cases:

i. The low dry-friction case of $F_{fk} = 15$ N;

ii. The high dry-friction case of $F_{fk} = 45$ N.

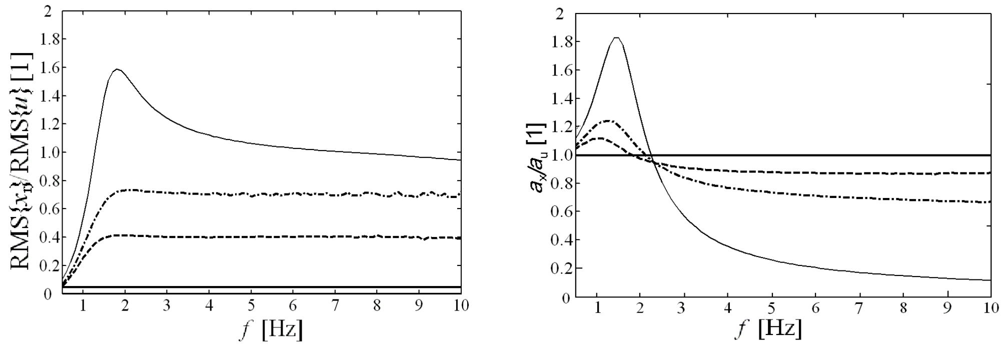
Due to inherent non-linearity the standard approach via the frequency response function calculation for assumed harmonic excitation with constant acceleration amplitude is not viable. Instead the ratio of response RMS value $a_x = \mathrm{RMS}\{\ddot{x}\}$ and excitation RMS value $a_u = \mathrm{RMS}\{\ddot{u}\}$ for excitation acceleration RMS constant values $a_u = 0.50\ \mathrm{m{\cdot}s^{-2}}$, $0.75\ \mathrm{m{\cdot}s^{-2}}$ and $1.00\ \mathrm{m{\cdot}s^{-2}}$ are calculated. A harmonic excitation is used with frequency step of 0.1 Hz in the frequency band 0.5 Hz till 10.0 Hz and the results are graphically depicted for each frequency step in Fig. 3 for the low friction system of $F_{fk} = 15$ N and in Fig. 4 for the high friction system of $F_{fk} = 45$ N, respectively. The left figures a) are the relative displacement transmissibility functions ($\mathrm{RMS}\{x_r\}/\mathrm{RMS}\{u\}$). The right figures b) are the acceleration transmissibility curves. For reference the same courses for a viscously damped SDOF oscillator without dry-friction are depicted too. The physically correct state switching model is used. Code is written in MATLAB/Simulink®, integration step is 2.5×10^{-3} s.

First of all the difference to a SDOF course without friction at frequencies above $\sqrt{2}\cdot f_0$ is seen in the right pictures – in dependence on the excitation amplitude the value of the transmissibility is larger than expected and so the vibration mitigation is less. For the higher friction case for low excitation amplitude the transmissibility value hovers around unity. This indicates, that the oscillatory system is in standstill (e.g. for $a_u \leq 0.50\ \mathrm{m{\cdot}s^{-2}}$) and so no vibration mitigation occurs.



a) b)

Figure 3 – Courses of transfer functions for SDOF oscillatory system with F_{fk} = 15 N for three excitation intensities and reference F_{fk} = 0 N (——); a_u = 0.50 m·s^{-2} (——); a_u = 0.75 m·s^{-2} (– – –); a_u = 1.00 m·s^{-2} (–·–·–)



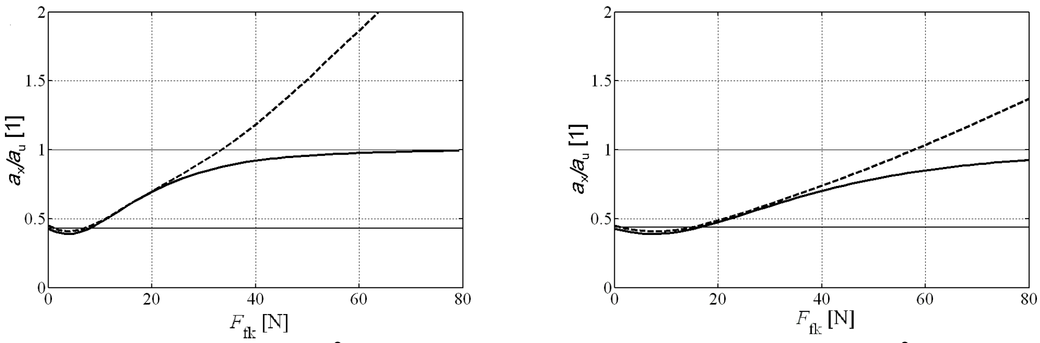
a) b)

Figure 4 – Courses of transfer functions for SDOF oscillatory system with F_{fk} = 45 N for three excitation intensities and reference F_{fk} = 0 N (——); a_u = 0.50 m·s^{-2} (——); a_u = 0.75 m·s^{-2} (– – –); a_u = 1.00 m·s^{-2} (–·–·–)

Relative displacement transmissibility at higher frequencies does not reach the expected unity value and for $a_u \leq 0.50$ m·s^{-2} hovers at zero, again indicating system standstill. The acceleration amplification at damped natural frequency f_d decreases due to friction influence. The decrease in amplification (increase in damping) is excitation amplitude dependent. These system features were observed while analysing signals from laboratory measurements of a seat suspension system, reported in [12].

PERFORMANCE UNDER REALITY-LIKE RANDOM EXCITATION

A problem, associated with zero crossing detection, is the numerical stability of the method utilising either signum function description in the vicinity of the zero crossing point. It has been observed, that the numerical solution exhibits false parasitic oscillations with a period $2\Delta t$, even if the real system would stop due to friction. Hence the non-linearity introduced by the signum function requires some care in application. The error appears mostly in the system output acceleration, especially when the system is subjected to low intensity random excitation. This is illustrated in Fig. 5 for two different stationary random excitation signals with $a_u = 0.33$ m·s^{-2} and $a_u = 0.67$ m·s^{-2}, representing reality-like excitation in response to variation of the kinetic friction force F_{fk}. The ratio of response and excitation RMS values is plotted for both simulation approaches, as well as for the viscously damped SDOF oscillator without dry-friction, which is obviously independent of the dry-friction influence.



a) $a_u = 0.33$ m·s^{-2} b) $a_u = 0.67$ m·s^{-2}

Figure 5 – Acceleration RMS values ratios for the SDOF oscillatory system $F_{fk} = 0$ N (——) and the two simulation approaches as function of F_{fk}: the signum approach (– – –), the slip-stick approach (—)

From these courses it is clearly seen, that the signum approach is not applicable if higher dry-friction values can be reasonably expected. The differences are due to the above-described parasitic oscillations. It is also seen, that the oscillatory system comes to rest when the dry friction values is increased and this state is excitation intensity dependent, which is not reflected at all either by the SDOF model without friction or the signum model. Fig. 5 indicates, that at low dry-friction values the SDOF system without dry-friction shows a marginally worst performance than both models including dry-friction. It predicts a slightly higher transmissibility course.

CONCLUSION

Two approaches to dry-friction modelling were presented and compared to each other and to the common SDOF oscillatory system. It has been shown that the physically correct approach, catering for switching between the slip and stick states, is of general nature. It describes the system performance for a wider dry-friction force range than the simpler signum approach does and caters for the stick phenomena. This is of prime importance if random kinematical excitation is used, as encountered in many practical situations. The difference in system performance to a common SDOF oscillator without dry-friction force influence has been demonstrated.

The presented approach is of generic nature and can be used for systems simulation with dry-friction, exposed to random kinematic excitation.

ACKNOWLEDGEMENT

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