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A PARTITION OF UNITY FORMULATION FOR THE CONVECTED WAVE EQUATION IN AXISYMMETRIC UNBOUNDED DOMAINS

Tanguy Mertens*[1], Philippe Bouillard[1], Jeremy Astley[2], Pablo Gamallo[2] and Laurent Hazard[1]

[1]Structural and Material Computational Mechanics Department,
Université Libre de Bruxelles,
Av. F. D. Roosevelt 50, CP 194/5, 1050 Brussels, Belgium
[2]Institute of Sound and Vibration Research, University of Southampton,
Highfield Road, Southampton SO17 1BJ – UK
email: Philippe.Bouillard@ulb.ac.be

Abstract
In this paper, an Infinite Partition of Unity Method (IPUM) is implemented to study axisymmetric applications. The formulation is based on the convected wave equation taking into account the effect of non-uniform flows on acoustic propagation. The concept of Infinite Elements developed by Astley et al. is used to deal with unbounded regions. The domain is first decomposed in an inner and an outer region. The PUM is used in the inner region while the outer one is discretized by a finite number of Partition of Unity Infinite Elements. The particularity of the PUM is that the shape functions can be constructed such that they correspond to a better approximation of the solution.

INTRODUCTION

During the last few years, the acoustic aspect has become a new design criteria because of restrictive standards and because of the need for increased comfort for customers. Computational methods are used to predict and minimise the noise of new products.

A common approach to simulate acoustic propagation is the use of the Finite Element Method (FEM). This is an efficient technique while the excitation frequency is kept moderate. The FEM is a deterministic approach. The FE mesh has to be generated such that the waves can be represented accurately by the shape functions. This means that the mesh will depend on the frequency. In general, a rule of the thumb is considered : 6 to 10 elements are needed to approximate a wavelength.

This rule is often used because it is practical. However, it has been shown [1] that this rule is not valid for medium and high frequencies. Ihlenburg showed that the error (in relative H_1 semi-norm) of the acoustic finite element solution is composed of two terms : the interpolation (1) and the pollution error (2).

$$\varepsilon_{|H_1} = \underbrace{C_1(Kh)^p}_{(1)} + \underbrace{C_2K(Kh)^{p+1}}_{(2)} \quad (Kh < 1) \tag{1}$$

where h is the size of the mesh, p is the polynomial order of the shape functions and K is the non-dimensional wavenumber ($K= 2\pi fL/c$), f being the excitation frequency, L a characteristic length and c, the sound speed.

Equation 1 shows that keeping the term Kh constant while increasing the frequency is not sufficient to ensure a constant error level. The effect of this relation is that the mesh should contain more than the classic 6 to 10 elements per wavelength. Due to the pollution error, when the excitation frequency increases, the mesh has to be deeply refined in order to ensure the accuracy of the numerical solution.

The Partition of Unity Method is explored in this paper in order to overcome this limitation. Our concern corresponds in convected acoustic propagation for radiating applications. This is the case while considering turbofan noise.

The influence of non uniform flow on acoustic propagation is taken into account in the formulation through the convected wave equation. This equation is obtained by assuming a potential flow. Gamallo [2,3] studied convected propagation with the Partition of Unity Method considering a plane wave enrichment.

Dealing with radiating applications imposes to consider infinite domains. Infinite Elements enable to mesh an infinite domain with a finite number of elements but fail to provide accurate solution close to the boundary conditions. Practically, the domain will be subdivided in an inner region and an outer one. The Mapped Wave Envelope Infinite Element (MWEIE) developed by Astley [4] is considered in this paper. Astley et al. [5] proposed the use of Legendre polynomials for the radial basis instead of the original Lagrange polynomials, Dreyer [6,7] suggested the use of Jacobi polynomials and Eversman [8] extended Infinite Elements to the convected case.

CONVECTED WAVE FORMULATION

This section describes how to obtain a scalar equation corresponding to the convected wave propagation. This equation comes from the mass conservation, the momentum equation, some thermodynamic relations and the assumption of a compressible inviscid isentropic irrotational flow.

Convected Wave Equation

The development of the convected wave formulation assume that the fluid is an ideal gas, that it is non viscous and that it does not conduct heat. We assume the flow is uniform at large distance and stationary. The influence of the gravity forces is neglected. We also assume that fluid elements are in thermodynamic equilibrium. These assumptions leads to an irrotational flow everywhere.

All fields are decomposed in their steady mean and acoustic parts ($p = p_0 + p_{a,t}$). The acoustic part corresponds to small harmonic perturbations ($p_{a,t} = p_a e^{i\omega t}$ with $i^2 = -1$). We also take into account the principal assumption which links the velocity field $\mathbf{v}$ to the potential (ϕ) : $\mathbf{v} = \nabla\phi$. The mass equation written in terms of potential leads to two equations. Grouping the zero order terms leads to equation 2 which has to be solved to provide the steady mean flow : c_0, ρ_0 and $\mathbf{v}_0$. Manipulations on the first order terms of the mass equation allow to write the convected wave equation which has the advantage to be scalar (eq. 3); ρ is the density. The pressure p is linked to the potential through the relation (4).

$$\nabla\cdot(\rho_0\mathbf{v}_0) = 0 \tag{2}$$

$$\left[\nabla\cdot(\rho_0\nabla\phi_a) - \nabla\cdot\left(\frac{\rho_0}{c_0^2}(\mathbf{v}_0\cdot\nabla\phi_a)\right)\right] - i\omega\left[\frac{\rho_0}{c_0^2}(\mathbf{v}_0\cdot\nabla\phi_a) + \nabla\cdot\left(\frac{\rho_0}{c_0^2}\mathbf{v}_0\phi_a\right)\right] + \omega^2\frac{\rho_0}{c_0^2}\phi_a = 0 \tag{3}$$

$$p_a = -\rho(i\omega\phi_a + \mathbf{v}_0\cdot\nabla\phi_a) \tag{4}$$

The variational formulation is obtained by using a standard weighted residual procedure for equation 3 and by integrating it by parts over the domain Ω; S being the boundary, W the weight function and V a Sobolev space.

$$\begin{aligned}
&\int_\Omega(\rho_0\nabla\phi_a)\cdot\nabla W\, d\Omega - \int_\Omega\nabla W\cdot\left(\frac{\rho_0}{c_0^2}\mathbf{v}_0(\mathbf{v}_0\cdot\nabla\phi_a)\right)d\Omega + i\omega\int_\Omega\frac{\rho_0}{c_0^2}(\mathbf{v}_0\cdot\nabla\phi_a)W\, d\Omega\\
&\quad - i\omega\int_\Omega\nabla W\cdot\left(\frac{\rho_0}{c_0^2}\mathbf{v}_0\phi_a\right)d\Omega - \omega^2\int_\Omega\frac{\rho_0}{c_0^2}W\phi_a\, d\Omega\\
&\quad = \int_S(W\rho_0\nabla\phi_a)\cdot\mathbf{n}\, dS - \int_S\frac{\rho_0}{c_0^2}W\mathbf{v}_0(\mathbf{v}_0\cdot\nabla\phi_a)\cdot\mathbf{n}\, dS\\
&\quad - i\omega\int_S\left(W\frac{\rho_0}{c_0^2}\mathbf{v}_0\phi_a\right)\cdot\mathbf{n}\, dS \qquad \forall W\in V
\end{aligned} \tag{5}$$

Partition of Unity Method

The Partition of Unity Method has been proposed by Melenk and Babuška [9]. The characteristic of this method comes from the way the approximation subspace is defined. The approximation (Φ) of the field (ϕ) is constructed on a mesh like conventional finite element model. The partition of unity method is based on the property (eq. 6) of partition of unity functions which enables the construction of a conforming space : $V_h \in V$. The approximation is given by equation 7 where V_{jl} are called enrichment functions or local approximation space and e_{jl} are the unknowns.

$$\sum_{j=1}^{nodes} N_j = 1 \tag{6}$$

$$\Phi_a = \sum_{j=1}^{nodes} N_j \sum_{l=1}^{dofs(j)} V_{jl} e_{jl} \tag{7}$$

Infinite Elements

The infinite elements allow to deal with exterior domains by discretizing it with a set of conventional elements within an inner region (R_i) and a set of infinite elements in an outer region (R_o). These two regions are separated through an interface Γ. The infinite elements are created from the nodes of the conventional mesh lying on the interface. Figure 1 shows the construction of infinite elements. It also illustrates that infinite elements are mapped on a square parent element. All further integrations will be done on this parent element.

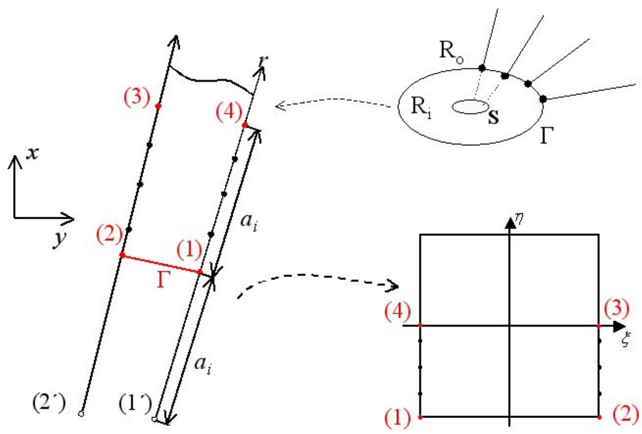

Figure 1 – Topology of an infinite element and its parent element

The aim of the infinite elements is to represent the radiation of the wave in the outer region without reverberation when the wave goes through the interface. This is done by assuming that the decay in the infinite field behaves like the radiation of a multipole.

The presence of the inner region is threefold :

- it has to represent the field until the interface where we assume the field behaves like the radiation of a multipole;
- proper infinite elements have to be created on a convex interface, this is rarely the case for the geometries of industrial applications;
- an other condition is that the flow has to be uniform in the outer region.

Formulation

The inner region is governed by the system previously developed for the convected wave equation. The same integral relations can be used in the outer region if the Sommerfeld radiation condition (eq. 8) is taken into account. This equation ensures that the energy propagates in an outward direction. The shape and weight functions are chosen such that the contribution of the Sommerfeld equation vanishes. In this case, there is no contribution of the surface at infinity.

$$r\left\{\frac{\partial p}{\partial r} + ikp\right\} \rightarrow 0 \qquad for \quad r \rightarrow \infty \tag{8}$$

The infinite shape function at node j (eq. 9) is composed by an interpolant function and a wavelike factor. The interpolant function takes into account a tangential (Q) and a radial shape function (eq. 9); η being the coordinate in the parent element corresponding to the infinite radial direction.

$$\Phi_{aj}(\mathbf{x},\omega) = P_j(\mathbf{x})e^{-ik\mu(\mathbf{x})} \qquad P_j(\mathbf{x}) = Q(\mathbf{x})\left(\frac{1-\eta}{2}\right)L_j^m(\eta) \tag{9}$$

The tangential shape function will be chosen to ensure the potential to be continuous across the interface Γ. The radial shape functions express the decay of the amplitude. Radial functions of initial infinite elements [4] were based on Lagrange polynomials. An expansion of this type contains a radial basis for spherical Bessel functions. In this work, Lagrange polynomials have been replaced by shifted Legendre ones to improve the conditioning.

The wavelike factor express the fact that the potential propagates with a wavenumber k. The phase function takes into account the influence of the uniform flow and is defined by equations 10 where M_0 is the Mach number, (x', y') are the coordinates of source points while (x, y) are the coordinates of nodes lying on the interface (fig. 1).

$$\begin{aligned} \mu(\mathbf{x}) &= \Psi(\mathbf{x}) - \Psi_1 \qquad \Psi_1 = \frac{1}{1-M_0^2}\left[-M_0(x-x') + R_1\right] \\ \Psi(\mathbf{x}) &= \frac{2\Psi_1}{1-\eta} \qquad R_1 = \sqrt{(x-x')^2 + \left(1-M_0^2\right)(y-y')^2} \end{aligned} \tag{10}$$

The weight functions are chosen following a conjugated (Petrov) Galerkin scheme. They are taken as complex conjugate of the shape functions times a geometric factor D. The geometric factor is chosen such that the integral over ∞ vanishes. It could be noticed that the weight function is taken as the conjugate of the shape function. This choice ensures the cancellation of the exponential terms in the weak formulation and leads to an easier integration.

Coupling

Since the acoustic potential must have continuity C_0 through the inner and the outer region and because the degrees of freedom on the interface Γ stand for the two regions, the shape functions of each region must have the same value on the interface. Since on the interface, $\eta=-1$ and $\mu=0$, the far field shape functions on the interface depend only on the tangential part : $Q(\mathbf{x})$. This means that on the interface, the tangential part of the infinite shape functions have to be the same than the near field shape functions. Since the near field is discretized by partition of unity, so does the tangential shape functions.

RESULTS

Two convergence analysis have been performed to evaluate the performance of the method. In the first study, we consider the propagation of a wave in a convected axisymmetric duct; in the second, we model the radiation of a dipole. The convergence curves compare relative errors (eq. 11) where p_a and p_c are, respectively, the analytic and computed pressures.

$$\varepsilon_R = 100 \frac{\sqrt{\int (p_a - p_c)^2}}{\sqrt{\int p_a^{\,2}}} \% \qquad (11)$$

For the problem 1, the duct has a radius of 0.5m and is 2m long; the flow is propagating in the opposite direction to the wave with a Mach number of 0.4. The excitation frequency is equal to 600Hz and the propagating mode is the first mode of the second axisymmetric order.

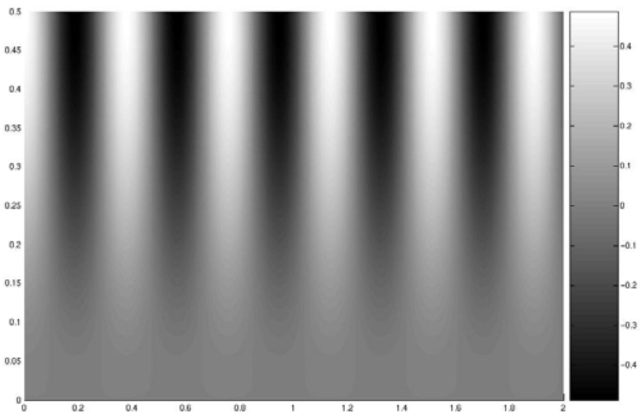

Figure 2 – Axisymmetric view of the real part of the pressure : p_c (6192 dofs)

The convergence curves (fig. 3) compare both linear FEM and PUM results. The Partition of Unity enrichment basis is defined by $V_{jl} = \{1 \quad x^2 \quad y^2\}$. The convergence is obtained by increasing proportionally the number of elements along both *x* and *y* directions. We see that the PUM convergence is faster than for the FEM (fig. 3). PUM relative error (%) is lower and its convergence rate is higher : the FEM convergence rate is proportional to h^2 while the PUM one behaves like h^4.

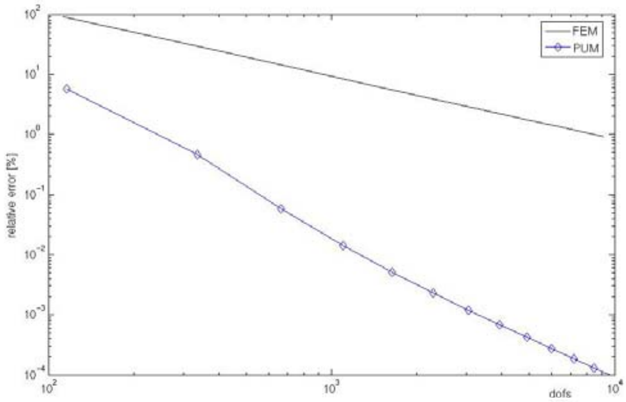

Figure 3 – Convergence curves : relative error [%] versus dofs

The second problem concerns a circular vibrating surface with a radius *R*=a, radiating in an outer region. The circular interface *Γ* at a distance *R*=b separates the infinite domain in an inner and an outer region. The axisymmetric order is 1 and the excitation frequency equals 400 Hz. The relative error is obtained through an integration over the inner mesh.

We show (fig. 4-a) that the convergence rate for linear IFEM depends on the radial order of the infinite elements. However, the convergence rate is slower than the expected one (h^2 for cavities) even for high radial order (IE=20). While comparing IPUM and IFEM (IE=10) (fig. 4-b) we show that IPUM is more accurate than IFEM. As seen before, the partition of unity discretization (in the inner region : *R*<*b*) leads to a better convergence rate. But because of the coupling, the final convergence is restricted by the influence of the infinite elements.

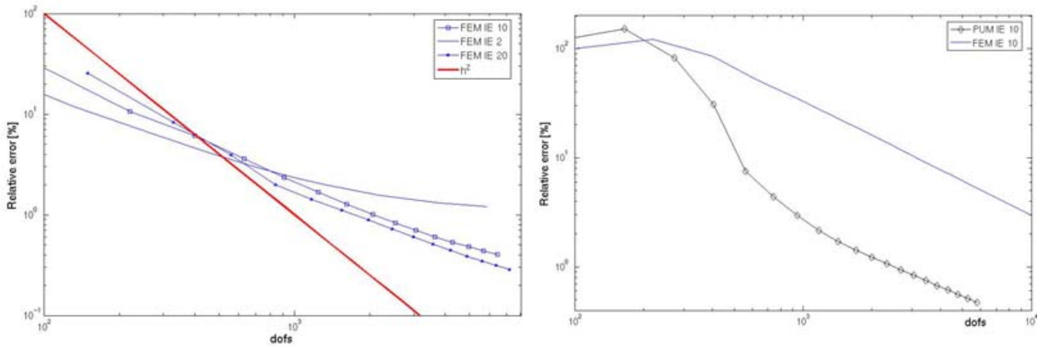

Figure 4 – Convergence curves : relative error [%] versus dofs
a) Linear FEM with IE order 2-10-20 and rate h^2 (a=1m; b=2m)
b) Linear FEM and PUM with IE order 10 (a=1m; b=4m)

CONCLUSIONS

This paper has presented a partition of unity formulation for convected radiation. The convected formulation and the coupling with infinite elements have been detailed. Two applications were shown to compare the finite element and the partition of unity methods.

Good convergence results were obtained with PUM enrichment, but the performance for radiation problem is still limited by the order of the infinite elements, as it is in FEM.

The global aim is to improve aircraft engine acoustics. Some developments have still to be done to consider axisymmetric nacelles. Further work will also include an optimisation procedure to find out the appropriate liner value.

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