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# IDENTIFICATION OF FRACTIONAL-DERIVATIVE-MODEL PARAMETERS OF VISCOELASTIC MATERIALS USING AN OPTIMIZATION TECHNIQUE

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**Abstract**

Viscoelastic damping materials are widely used to reduce noise and vibration of structures because of its low cost and easy implementation. To design the damped structures, the material property such as elastic modulus and loss factor is essential information. The four-parameter fractional derivative model well describes the nonlinear dynamic characteristics of the viscoelastic damping materials than conventional spring-dashpot models. However, the identification procedure of the four parameters is very time-consuming process. In this study, an efficient identification procedure of the four parameters is proposed by using an FE model and a gradient-based numerical search algorithm. The identification procedure goes two sequential steps to make measured frequency response functions (FRF) coincident with simulated FRFs: the first one is a peak alignment step and the second one is an amplitude adjustment step. A numerical example shows that the proposed method is efficient and robust in identifying the fractional-derivative-model parameters of viscoelastic materials.

## INTRODUCTION

Damping materials are widely used to control passively sound and vibration problems of structures because of its low cost and easy implementation [1]. For example, on the body structure of passenger cars there are many damping sheets in order to reduce the vibration of panels. The similar cases are found in airplanes, ships and electric appliances. To design damping layer layout of structures, an efficient model of the damped structures is needed to describe the dynamic characteristics of viscoelastic damping materials. Nowadays, it is well known that the four-parameter fractional derivative model is one of the best models for viscoelastic damping materials [2].

However, to obtain the four parameters of the fractional derivative model with conventional method, many tests are required at different frequencies and temperatures. Moreover, trial-and-error approach in data analysis makes the conventional parameter identification method very time-consuming. In this study, the authors propose an efficient and robust method to identify the parameters of the fractional derivative model of viscoelastic materials using a gradient-based numerical search algorithm.

## IDENTIFICATION OF THE MATERIAL PROPERTIES

### Fractional Derivative Model of Viscoelastic Materials

Dynamic characteristics of the viscoelastic materials in frequency domain can be represented using the complex modulus such as:

$$\bar{\sigma} = E^* \bar{\varepsilon} = E(1 + \boldsymbol{i}\eta)\bar{\varepsilon} \tag{1}$$

where $\boldsymbol{i} = \sqrt{-1}$, $\bar{\sigma}$ and $\bar{\varepsilon}$ are the Fourier transforms of stress and strain, respectively. $E^*, E$ and $\eta$ are the complex modulus, the storage modulus and the loss factor, respectively.

The complex modulus of viscoelastic materials is strongly dependent on temperature as well as frequency. However, we can predict the complex modulus at any temperatures using the shift factor $\alpha$(T) from the temperature-frequency superposition principle of viscoelastic materials. The shift factor is coupled with temperature through the Arrhenius equation such as [6]:

$$log(\alpha(T)) = d_1\left(1/T - 1/T_0\right) \tag{2}$$

where $d_1$ is a constant and $T_0$ is a reference temperature in degrees absolute.

Considering the frequency variation of damping behavior as well as temperature variation, the complex modulus of the fractional derivative model in frequency domain can be written as follows [5, 6].

$$E^* = E(1 + i\eta) = \frac{a_0 + a_1(if\alpha(T))^\beta}{1 + c_1(if\alpha(T))^\beta} \tag{3}$$

Here, the four parameters $a_0, a_1, c_1$ and $\beta$ in equation (3) are identified by a suitable empirical way.

It is well known that the four-parameter fractional derivative model is sufficient

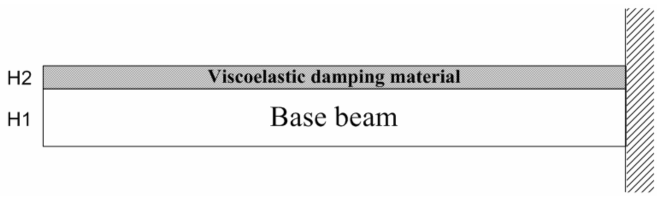



Figure 1. Oberst beam test configuration.

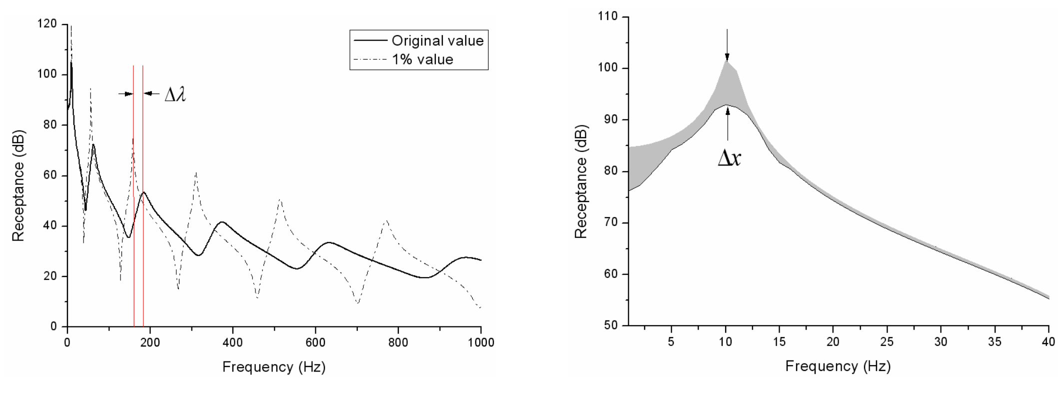
to represent the real behavior of viscoelastic materials over a wide frequency range [6]. Therefore, identifying the six parameters of a viscoelastic material, the fractional derivative model can describe the dynamic characteristics of the viscoelastic materials over frequency and temperature variations. To estimate the fractional-derivative-model parameters of a real material with conventional methods, first many tests should be repeated until sufficient number of data are acquired at different frequencies and temperatures using, for example, Oberst beam test as shown in Fig. 1. Second, from these data, the coefficients of the fractional derivative model can be determined using a statistical data analysis technique that minimizes the mean square error between theoretical value and the tabulated value [6]. However, the statistical data analysis process is not so efficient because it includes trial-and-error steps, i.e., the shift factor is assumed and the mean square error is minimized. The trial-and-error step is repeated in turn until the global error value is obtained.

## A New Identification Method

To develop a new estimation method of the fractional-derivative-model parameters, the authors start from an assumption that if a numerical model reproduces measured responses, then material properties used in the simulation model is the real material properties of the material. Then by minimizing the response difference between the measured and simulated FRFs, one can identify the material properties using a numerical search algorithm. The basic idea is adopted from the author's previous work [5]. In the previous work, the identification index function that is zero at the true values and should be minimized for the identification is defined as follows.

$$g(b) = \sum_{i=1}^{N} \int (x^i_{simulated} - x^i_{measured})^2 \, df \qquad (4)$$

Here, $x$, N and $f$ are frequency responses, number of responses and frequency, respectively. Generally, gradient-based mathematical programming techniques are used to minimize the identification index because the gradient-based methods are the




(a) Step 1 : Peak alignment step (b)Step 2 : Amplitude adjustment step

Figure 2. The two-step identification procedure

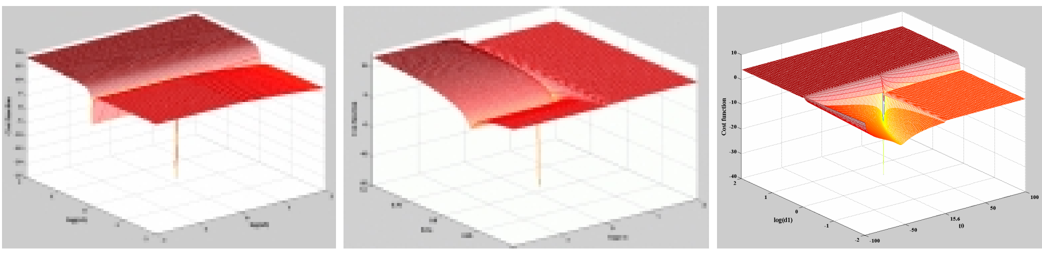
most efficient although it may give a local minimum.

The convex region of the identification index function should be as wide as possible in order that the identification procedure can give true values consistently regardless of initial values. In our previous work [5], the identification index defined in Eq. (4) sometimes fell into a local minimum if initial values far from the true values are given. To widen the stable region of the identification process, the authors introduce a new identification index and divide the process into two steps. The first step is a peak-alignment step and the second one is an amplitude-adjustment step as shown in Fig. 2. As a result, the identification index defined in Eq. (4) is split into two as follows:

$$g_1(b) = \sum_{i=1}^{N}\sum_{k=1}^{M}(\lambda^i_{k,simulated} - \lambda^i_{k,measured})^2 \quad (5)$$

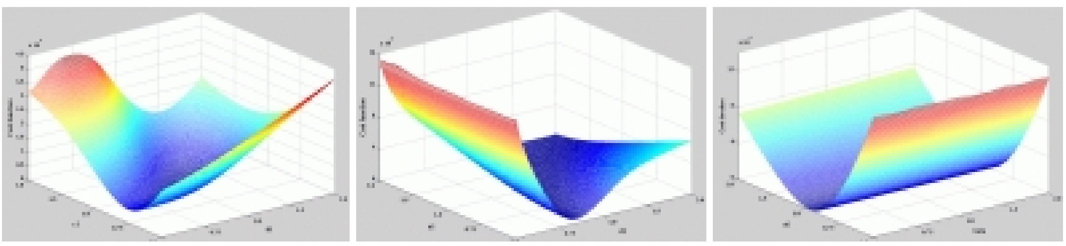
$$g_2(b) = \sum_{i=1}^{N}\int (x^i_{simulated} - x^i_{measured})^2 df \quad (6)$$

where $\lambda$ and M are resonance frequencies and the number of resonant peaks within a concerned frequency range, respectively. Then, minimizing the first identification index function with respect to the six parameters of the factional derivative model, the response differences will be very small. Therefore, the second step that is a minimization step of magnitude-difference between the measured and simulated FRFs, can be started from very close values to true values, which means that the identification process has little possibility of falling in a local minimum. Figures 3 and 4 show the contour surface of the identification index functions for a typical damped beam



(a) log( $a_0$ ) vs. log( $a_1$ )　(b) log( $c_1$ ) vs. $\beta$　(c) log( $d_1$ ) vs. $T_0$

Figure 3. Contour surfaces of the first-step identification index



(a) $c_1$ vs. $T_0$　(b) $a_1$ vs. $d_1$　(c) $a_0$ vs. $\beta$

Figure 4. Contour surfaces of the second-step identification index

problem according to the parameters normalized to the real values. As shown in the figures, the first identification index function is sufficiently smooth over the wide region, and the second identification index function is also very smooth near the true values.

Summarizing the proposed method, the identification procedure consists of two sequential stages. In the first stage, the resonant-frequency differences are minimized by using the identification index with arbitrary assumed values, Eq. (5). In the second stage Eq. (6) is used as an identification index and the minimization process started from the results of the first stage gives best-fitted parameters of the fractional derivative model. It should be noted here that the number of measured responses must be larger than two because a single frequency response at a temperature does not contain sufficient information of dynamic behavior due to temperature variations.

## ANALYSIS MODEL OF DAMPED BEAM

For the identification process a simulation model of the damped beam is necessary. In addition, the gradient information of the identification index function with respect to the unknown parameters should be provided in order to search minimum points using a numerical search algorithm.

### Finite element analysis of unconstrained damping layer beam

From the Ross, Ungar and Kerwin (RUK)'s equation, the equivalent complex flexural rigidity, $E^*I$, of the unconstrained plates is written in the form [7, 8]:

$$\frac{E^*I}{E_1^*I_1} = 1 + e^*h^3 + 3(1+h)^2 \frac{e^*h}{1+e^*h} \tag{7}$$

where $h = H_2/H_1$, $e^* = E_2^*/E_1^*$ and $I$ is the second area moment. From the RUK equation the equivalent storage modulus of the unconstrained beam is the real part of equation (7), and the equivalent loss factor can also be obtained from the imaginary part of equation (7).

Introducing a finite beam element that has flexural displacements and rotations, one can obtain the equations of motion such as [4]:

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{F} \tag{8}$$

where $\boldsymbol{M}$ and $\boldsymbol{K}$ are the global mass and stiffness matrices, respectively, and $\boldsymbol{x}$ and $\boldsymbol{F}$ are the displacement and force vectors, respectively. In addition, the complex-valued matrix $\boldsymbol{K}$ satisfies the following relation:

$$\boldsymbol{K} = \boldsymbol{K}_r + \boldsymbol{i}\boldsymbol{K}_i = \boldsymbol{K}_r(1 + \boldsymbol{i}\eta) \tag{9}$$

where the subscripts *r* and *i* mean the real and imaginary parts, respectively. Assuming a harmonic motion of the system, the corresponding real eigenvalue problem can be written as:

$$\boldsymbol{K}_r \boldsymbol{y} = \varsigma \boldsymbol{My} \tag{10}$$

where $\boldsymbol{y}$ is the eigenvector and $\varsigma (= \omega^2 = (2\pi f)^2)$ is the eigenvalue. The eigenvalue problem of equation (10) is nonlinear and the iteration procedure is summarized well in Ref [4].

The modal superposition principle gives an expression of harmonic responses in vibration problems and the displacement of the damped structure can be written as:

$$\boldsymbol{x} = \sum_{\boldsymbol{k}=1}^{\boldsymbol{m}} a_k \boldsymbol{y_k} \qquad \text{where} \quad a_k = \boldsymbol{y_k}^{\boldsymbol{T}} \boldsymbol{F} \Big/ \left( \zeta_{\boldsymbol{k}} (1 + \boldsymbol{i}\eta_{\boldsymbol{k}}) - \omega^2 \right) \tag{11}$$

Here, $m$ is the number of mode, $\boldsymbol{y_k}$ the $k$-th eigenvector, and $a_k$ the $k$-th modal coordinate. $\eta_{\boldsymbol{k}}$ is the loss factor of $k$-th mode and defined by energy ratio as [8]:

$$\eta_{\boldsymbol{k}} = \sum_{\boldsymbol{j}=1}^{\boldsymbol{n}} \eta_{\boldsymbol{ej}} \boldsymbol{U}_{\boldsymbol{ej}} \Big/ \sum_{\boldsymbol{j}=1}^{\boldsymbol{n}} \boldsymbol{U}_{\boldsymbol{ej}} = \sum_{\boldsymbol{j}=1}^{\boldsymbol{n}} \eta_{\boldsymbol{ej}} \boldsymbol{U}_{\boldsymbol{ej}} \Big/ \boldsymbol{U} \tag{12}$$

where $n$ is the number of finite elements, $\eta_{\boldsymbol{ej}}$ is the loss factor of the $j$-th element, $\boldsymbol{U}_{\boldsymbol{ej}}$ is the strain energy of the $j$-th finite element, and $U$ is the total strain energy. In this study, the real eigenvectors will be used to evaluate the loss factor.

### Parameter Sensitivity Analysis

To identify the six-parameters of the fractional derivative model using the gradient-based algorithms, the sensitivity analysis for the identification indexes are needed. The parameter sensitivity information can be obtained analytically by differentiating the identification index expressions with respect to the fractional-derivative-model parameters. The results sensitivity equation consists of eigenvalue and eigenvector sensitivities and derivative expression of the complex modulus represented by the fractional derivative model. The details of the parameter sensitivity analysis method can be found in Ref [3] and will not be repeated here for lack of space.

## NUMERICAL EXAMPLES

To validate the identification procedure, a numerical experiment is introduced. A known viscoelastic damping material, LD-400 [6], is bonded on an aluminum beam. The beam is modeled by 20 finite elements with equivalent stiffness and the point

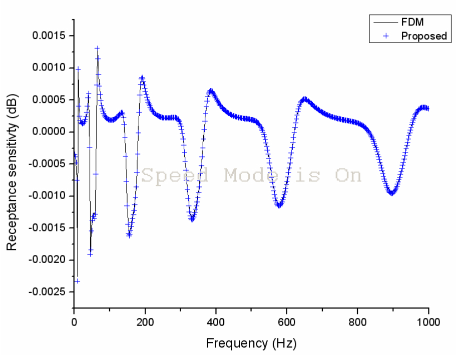



Figure 5. Parameter sensitivity w.r.t. $a_0$ compared with that of finite difference method

**Table 1. Identification Results**

| Results / Method | | No. of iter | Ratio to the true values (%) | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | $a_0$ | $a_1$ | $c_1$ | $\beta$ | $d_1$ | $t_0$ |
| **One step** | | **152** | **100** | **100** | **100** | **98** | **100** | **100** |
| **Two-step** | **1st** | **23** | **99** | **100** | **89** | **98** | **100** | **97** |
| | **2nd** | **41** | **100** | **100** | **100** | **99** | **100** | **100** |

receptance frequency responses at the free end point of the beam are calculated by the FE model. In this study, the FRFs calculated from the FE model with the true material parameters are used as the reference frequency responses for the identification problem. The reference FRFs are calculated at two difference temperatures, i.e., 20 $°C$ and 80 $°C$.

The six parameters of LD-400 are identified by the proposed method. To check the correctness of the gradient information, the parameter sensitivity calculated by the analytic formula is compared with that of finite difference method as shown in Figure 5. The calculated parameter sensitivity information is plugged into optimization software, DOT [9], to solve the inverse problem. Table 1 shows the identification results with the two-stage method compared with one-step strategy. One can see in Table 1 that the proposed method is very efficient than the one-step method. Next, to verify the robustness of the proposed method with respect to initial values, the identification process is repeated with different initial values from -2 to +2 orders of magnitude to the true values except parameter $\beta$. The order of fractional derivative, $\beta$, is always

Table 2. Identification results started from P-multiple values of the true values

| Initial Values | | (True Value/ Identified Value)×100 [%] | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | **P=0.01** | | **P=0.1** | | **P=10.** | | **P=100.** | | **P=random** | | **P=±10** | |
| Para. | True Value | **1st** | **2nd** | **1st** | **2nd** | **1st** | **2nd** | **1st** | **2nd** | **1st** | **2nd** | **1st** | **2nd** |
| $a_0$ | 338.2 | **99.93** | **100.3** | **100.1** | **100.1** | **101.8** | **99.99** | **100.0** | - | **98.27** | **99.99** | **100.0** | **100.0** |
| $a_1$ | 2485 | **100.0** | **100.0** | **100.2** | **100.0** | **99.19** | **100.0** | **101.9** | - | **100.8** | **100.0** | **100.8** | **100.0** |
| $c_1$ | 0.12 | **103.5** | **100.0** | **112.5** | **100.0** | **575.0** | **100.8** | **9957.** | - | **77.96** | **100.8** | **88.95** | **100.0** |
| $d_1$ | 12222 | **100.6** | **99.85** | **101.4** | **99.79** | **99.97** | **100.0** | **999.8** | - | **100.0** | **100.0** | **90.02** | **99.79** |
| $\beta$ | 0.47 | **99.99** | **100.0** | **100.0** | **100.0** | **99.76** | **99.99** | **106.3** | - | **100.2** | **99.99** | **100.2** | **99.99** |
| $T_0$ | 15.6 | **99.43** | **100.0** | **103.2** | **100.5** | **100.7** | **100.0** | **9943.** | - | **99.32** | **100.0** | **100.6** | **100.0** |

assumed as 0.5 initially. Table 2 shows the identified values. It is shown in Table 2 that the proposed method can identify the unknown material properties even with very rough initial assumptions. In addition, it should be noted that the possibility to reach on the true values is higher when started from smaller values to the true values than when started from larger values. Therefore, if the difference of response could not be minimized with an initial value, we should restart with smaller values than the current initial assumption in order to reach the true values.

## CONCLUSIONS

An efficient identification method of material parameters represented by the fractional derivative model is proposed using a gradient-based optimization technique. To identify the fractional-derivative-model parameters, the Oberst beam coated on one side by a viscoelastic material is modeled by finite beam elements. The elastic modulus and loss factors of the equivalent beam elements are obtained from Ross, Ungar and Kerwin's equation and the complex modulus expression of the fractional derivative model of the viscoelastic material. Then the frequency response functions on the same points with the measured one are calculated with assumed fractional-derivative-model parameters. The differences between the measured and the calculated FRFs are minimized by using a gradient-based optimization algorithm to identify the real values of the parameters. For efficient search iteration, the analytic gradient information is used. The difference minimization procedure consists of the peak-alignment stage and amplitude adjustment stage. Numerical experiments show that the proposed method accurately identifies the fractional-derivative-model parameters of viscoelastic materials even with a rough initial assumption of the parameters.

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