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# INTERVAL DYNAMIC ANALYSIS FOR STRUCTURES WITH BOUNDED UNCERTAIN PARAMETERS

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## Abstract

A new method called the interval factor method for determining the natural frequencies and modeshapes of truss structures with bounded uncertain parameters is presented in this paper. Using the interval factor method, the structural material parameters and geometric dimensions can be considered as interval variables, in which a structural parameter can be expressed as its mean value multiplied by its interval factor. Computational expressions for the lower and upper bounds and mean value of the natural frequencies and modeshapes are derived using the Rayleigh quotient. The influences of the uncertainties of the structural parameters on the dynamic characteristics of truss structures are investigated. The main advantage of using the interval factor method is that the effect of uncertainty of any individual structural parameter on the dynamic characteristics can be easily examined.

## INTRODUCTION

The natural frequency and mode shape analysis of structures with uncertain parameters is a very significant research field in engineering [1,2]. In most practical engineering cases, structures have uncertainty in their parameters arising from materials defects, manufacturing tolerances and variation in operating conditions.

The interval analysis method appeared in 1966 [3]. Moore [4] and Alefeld [5] have done the pioneering work, where they established the basic theory for the interval analysis and discussed the interval operations and its application. Recently, Chen *et al*. [6,7] and Qiu and Wang [8] have conducted eigenvalue analysis of structures with bounded uncertain parameters using interval perturbation method (IPM). In perturbation methods, the uncertainty of all the structural parameters are expressed as small parameters in the structural mass and stiffness matrices. These small parameters are not interval variables, but simply small values. Therefore, it is very difficult to investigate the effect of the uncertainty of the individual parameters on the natural frequencies by their method. In addition, IPM does not necessarily yield a conservative approximation, as the effect of neglecting the higher order terms is unpredictable. Furthermore, in order to safely remove the higher order terms, IPM is only applicable to analyses with small intervals [1].

In this paper, the free vibrational characteristics of a dynamic structure with uncertainty in its parameters is investigated, using a new method called the Interval Factor Method (IFM). Truss structures are used to illustrate examples of this method, in which the Young's modulus, mass density, bar's length and cross-sectional area are considered as interval variables. The procedure of the IFM is as follows. Firstly, a structural parameter is expressed as an interval factor multiplied by the mean value of this parameter. Secondly, the structural mass and stiffness matrices are respectively expressed as interval factors of the parameters multiplied by their deterministic values. Finally, using Rayleigh's quotient, the natural frequencies and modeshapes can be expressed as functions of these interval factors. Therefore, the effect of the uncertainty of any of the individual structural parameters on the natural frequencies and corresponding modeshapes can be easily observed.

## INTERVAL FACTOR

Assume that $I(R)$ denotes the sets of all closed real interval numbers. $X^I = [x^l, x^u]$ is a member of $I(R)$, where $x^l$ and $x^u$ denote the lower and upper values of $x$, respectively. $X^I$ can be usually written in the following form

$$X^I = [\bar{x} - \Delta x,\ \bar{x} + \Delta x],\quad \bar{x} = \frac{x^l + x^u}{2},\quad \Delta x = \frac{x^u - x^l}{2} \tag{1}$$

where $\bar{x}$ and $\Delta x$ denote the mean value of $X^I$ and the uncertainty in $X^I$, respectively. Hence, an arbitrary interval $X^I = [x^l, x^u]$ can be written as the sum of its mean value and its uncertain interval $\Delta X^I = [-\Delta x,\ \Delta x]$

$$X^I = \bar{x} + \Delta X^I \tag{2}$$

Equation (2) can also be expressed as

$$X^I = [\bar{x}(1 - \frac{\Delta x}{\bar{x}}),\ \bar{x}(1 + \frac{\Delta x}{\bar{x}})] = \bar{x}[1 - \frac{x^u - x^l}{2\bar{x}},\ 1 + \frac{x^u - x^l}{2\bar{x}}] \tag{3}$$

Here, we introduce $X_f^I = [x_f^l, x_f^u]$ that is a member of $I(R)$ and let

$$X_f^I = [x_f^l, x_f^u] = [1 - \frac{x^u - x^l}{2\bar{x}}, 1 + \frac{x^u - x^l}{2\bar{x}}] \tag{4}$$

Substituting equation (4) into equation (3) yields

$$X^I = X_f^I \bar{x} \tag{5}$$

Since $\bar{x}$ is a deterministic value and the uncertainty of $X^I$ is described by $X_f^I$, $X_f^I$ is named as the interval factor of $X^I$ in this study and can be easily obtained by the following expressions:

$$x_f^l = 1 - \frac{x^u - x_l}{2\bar{x}}, \quad x_f^u = 1 + \frac{x^u - x_l}{2\bar{x}}, \quad \bar{x}_f = \frac{x_f^l + x_f^u}{2} = 1, \quad \Delta x_f = \frac{x_f^u - x_f^l}{2} = \frac{x^u - x^l}{2\bar{x}} = \frac{\Delta x}{\bar{x}} \tag{6}$$

where $\bar{x}_f$ denotes the mean value of $X_f^I$ and $\Delta x_f$ denotes the uncertainty in $X_f^I$. $\Delta x_f$ also denotes the interval ratio of $X^I$, corresponding to the ratio of the maximum uncertainty in $X^I$ to its mean value.

## INTERVAL NATURAL FREQUENC Y AND MODE SHAPE ANALYSIS

Suppose that there are $n$ elements in the truss structure under consideration. The mass matrix $[M]$ and stiffness matrix $[K]$ of the truss structure in global coordinates can be respectively expressed as

$$[M] = \sum_{e=1}^{m} [M_e] = \sum_{e=1}^{m} \frac{1}{2} \rho_e A_e L_e [I] \tag{7}$$

$$[K] = \sum_{e=1}^{m} [K_e] = \sum_{e=1}^{m} [T_e]^T \frac{E_e A_e}{L_e} [G][T_e] \tag{8}$$

where $[M_e]$ is the mass matrix of the $e^{\text{th}}$ element, $[K_e]$ is the stiffness matrix of the $e^{\text{th}}$ element. $E_e$, $A_e$, $L_e$ and $\rho_e$ are the Young's modulus, cross-sectional area, length and mass density respectively of the $e^{\text{th}}$ element. $[I]$ is a $6^{\text{th}}$ order identity matrix, $[G]$ is a $6 \times 6$ matrix ,where $g_{11} = g_{44} = 1$, $g_{14} = g_{41} = -1$, and all other elements of $[G]$ are equal to zero [9]. $[T_e]$ is a transformation matrix that translates the local coordinates of the $e^{\text{th}}$ element to global coordinates [9], and $[T_e]^T$ is its transpose.

In the following analysis, we consider the material parameters ( $\rho_e$ , $E_e$ ) and geometric dimensions ( $A_e$ , $L_e$ ) to simultaneously be members of $I(R)$, that is, they are all interval variables. Since the bars of the truss structure are of the same material, and all elements are manufactured in a similar way to each other, it is assumed that the interval ratio $\Delta x / \bar{x}$ (ratio of the maximum uncertainty to the mean value) of a given parameter for each element are the same. For example, for the Young's modulus, we have $\Delta E_1 / \bar{E}_1 = \Delta E_2 / \bar{E}_2 = ... = \Delta E_n / \bar{E}_n = \Delta E / \bar{E}$. The Young's modulus $E_e^I$, mass density $\rho_e^I$, cross-sectional area $A_e^I$ and length $L_e^I$ of the each element can respectively be expressed as interval variables by:

$$E_e^I = E_f^I \overline{E}_e, \quad \rho_e^I = \rho_f^I \overline{\rho}_e, \quad A_e^I = A_f^I \overline{A}_e, \quad L_e^I = L_f^I \overline{L}_e \tag{9}$$

where $E_f^I$, $\rho_f^I$, $A_f^I$ and $L_f^I$ are the interval factors of $E_e^I$, $\rho_e^I$, $A_e^I$ and $L_e^I$, respectively, and $\overline{E}_e$, $\overline{\rho}_e$, $\overline{A}_e$ and $\overline{L}_e$ are mean values of $E_e^I$, $\rho_e^I$, $A_e^I$ and $L_e^I$, respectively. By means of equation (6), the interval ratio expressions for the Young's modulus, density, cross-sectional area and length can be respectively expressed as

$$\Delta E_f = \frac{\Delta E}{\overline{E}}, \quad \Delta\rho_f = \frac{\Delta\rho}{\overline{\rho}}, \quad \Delta A_f = \frac{\Delta A}{\overline{A}}, \quad \Delta L_f = \frac{\Delta L}{\overline{L}} \tag{10}$$

where $\Delta E_f$, $\Delta\rho_f$, $\Delta A_f$ and $\Delta L_f$ are the interval ratios of $E_e^I$, $\rho_e^I$, $A_e^I$ and $L_e^I$, respectively.

From equation (7), the mass matrix of the $e^{\text{th}}$ element can be easily obtained in terms of its interval variables by

$$[M_e]^I = \frac{1}{2}\rho_f^I A_f^I L_f^I \overline{\rho}_e \overline{A}_e \overline{L}_e [I] = \rho_f^I A_f^I L_f^I [M_e]^{\det} \tag{11}$$

where $[M_e]^{\det}$ is the deterministic part of mass matrix $[M_e]^I$. It should be noted that $[M_e]^{\det}$ becomes the mean value of $[M]^I$ when only one of structural parameters is interval variable. Equation (11) shows that the mass matrix $[M_e]^I$ can be divided into the product of two parts, corresponding to the interval factors $\rho_f^I$, $A_f^I$, $L_f^I$ and the deterministic matrix $[M_e]^{\det}$. Constructing the deterministic matrix $[M_e]^{\det}$ is the same as constructing the mass matrix in equation (7) for the $e^{\text{th}}$ element, and taking the parameters as $\rho_e^I = \overline{\rho}_e$, $A_e^I = \overline{A}_e$ and $L_e^I = \overline{L}_e$. It is important to note that the mass matrix of the $e^{\text{th}}$ element needs to be expanded before assemblage. $[M]^I$ can now be written as:

$$[M]^I = \sum_{e=1}^{n}[M_e]^I = \sum_{e=1}^{n}\rho_f^I A_f^I L_f^I [M_e]^{\det} = \rho_f^I A_f^I L_f^I [M]^{\det} \tag{12}$$

where $[M]^{\det}$ is the deterministic part of the mass matrix $[M]^I$.

Similarly, the stiffness matrix can also be expressed in terms of its interval variables by:

$$[K]^I = \sum_{e=1}^{n}[K_e]^I = \sum_{e=1}^{n}\frac{E_f^I A_f^I}{L_f^I}[K_e]^{\det} = \frac{E_f^I A_f^I}{L_f^I}[K]^{\det} \tag{13}$$

where $[K_e]^{\det}$ and $[K]^{\det}$ are the deterministic part of $[K_e]^I$ and $[K]^I$, respectively.

Suppose that $j^{\text{th}}$ order natural frequency and modeshape of the structure are denoted $\omega_j{}^I$ and $\{\phi_j\}^I$, respectively. Using the interval factor method, $\omega_j{}^I$ and $\{\phi_j\}^I$ can be written as

$$\omega_j{}^I = \omega_{j\,f}{}^I \omega_j{}^{\det}, \quad \{\phi_j\}^I = \phi_{j\,f}{}^I \{\phi_j\}^{\det} \tag{14}$$

where $\omega_j^{\det}$ and $\{\phi_j\}^{\det}$ are the deterministic parts of $\omega_j{}^I$ and $\{\phi_j\}^I$, respectively. It is important to note that the deterministic components do not correspond to the mean values.

Substituting equations (12), (13) and (14) into Rayleigh's quotient [9] yields

$$(\omega_j{}^I)^2 = \frac{\{\phi_j\}^{I\,T}[K]^I\{\phi_j\}^I}{\{\phi_j\}^{I\,T}[M]^I\{\phi_j\}^I} = \frac{\phi_{j\,f}{}^I E_f^I A_f^I \phi_{j\,f}{}^I}{\phi_{j\,f}{}^I \rho_f^I A_f^I L_f^I L_f^I \phi_{j\,f}{}^I} \frac{\{\phi_j\}^{\det T}[K]^{\det}\{\phi_j\}^{\det}}{\{\phi_j\}^{\det T}[M]^{\det}\{\phi_j\}^{\det}}$$

$$= \frac{E_f^I}{\rho_f^I L_f^I L_f^I} \frac{K_j^{\det}}{M_j^{\det}} = \frac{E_f^I}{\rho_f^I L_f^I L_f^I} (\omega_j^{\det})^2 \tag{15}$$

$K_j^{\det}$, $M_j^{\det}$, $\omega_j{}^{\det}$ are all deterministic quantities, corresponding to the $j^{\text{th}}$ order stiffness, mass and natural frequency of the structure when the parameters are $E_e^I = \overline{E}_e$, $\rho_e^I = \overline{\rho}_e$, $A_e^I = \overline{A}_e$ and $L_e^I = \overline{L}_e$. From $K^{\det}$ and $M_j^{\det}$, $\omega_j{}^{\det}$ can be obtained from a conventional finite element model.

From equation (15), the interval values of $\omega_j{}^I$ can be obtained, corresponding to the lower and upper bounds and the mean value of the $j^{\text{th}}$ natural frequency, which are respectively given by:

$$\omega_j^l = \left(\frac{1-\Delta E_f}{(1+\Delta\rho_f)(1+\Delta Lf)(1+\Delta L_f)}\right)^{1/2} \cdot \omega_j^{\det} \tag{16}$$

$$\omega_j^u = \left(\frac{1+\Delta E_f}{(1-\Delta\rho_f)(1-\Delta L_f)(1-\Delta L_f)}\right)^{1/2} \cdot \omega_j^{\det} \tag{17}$$

$$\overline{\omega}_j = (\omega_j^l + \omega_j^u)/2 \tag{18}$$

From modal analysis theory, the modal matrix $[\phi]$ has the following orthogonal property:

$$[\phi]^T[M][\phi] = [I] \tag{19}$$

$$[\phi]^T[K][\phi] = [\Omega] = diag[\omega^2] \tag{20}$$

Equations (19) and (20) can be written as

$$\phi_f^I \rho_f^I A_f^I L_f^I L_f^I \phi_f^I ([\phi]^{\det T} [M]^{\det} [\phi]^{\det}) = [I] \tag{21}$$

$$\frac{\phi_f^I E_f^I A_f^I \phi_f^I}{L_f^I} ([\phi]^{\det T} [K]^{\det} [\phi]^{\det}) = \frac{E_f^I}{\rho_f^I L_f^I L_f^I} diag[(\omega^{\det})^2] \tag{22}$$

From equations (21) and (22), we can obtain the following expression for the interval factor for the modeshape:

$$\phi_{j\,f}^{\,I} = \phi_f^I = (\frac{1}{\rho_f^I A_f^I L_f^I})^{1/2} \tag{23}$$

Equation (23) states that the interval change ratio of each element in the modal matrix are equal. The interval values (lower bound, upper bound, mean value) of any elements $\phi_{ij}^{\ I}$ in the modal matrix can be obtained according to the interval operations:

$$\phi_{ij}^l = \left[1/((1+\Delta\rho_f)(1+\Delta A_f)(1+\Delta L_f))\right]^{1/2} \cdot \phi_{ij}^{\det} \tag{24}$$

$$\phi_{ij}^u = \left[1/((1-\Delta\rho_f)(1-\Delta A_f)(1-\Delta L_f))\right]^{1/2} \cdot \phi_{ij}^{\det} \tag{25}$$

$$\bar{\phi}_{ij} = (\phi_{ij}^l + \phi_{ij}^u)/2 \tag{26}$$

## NUMERICAL EXAMPLE

An 8-meter caliber antenna shown in Figure 1 is used as an example. The antenna is a 96-node and 336-element space truss structure, with 12 elements. The mean values of the cross-sectional area of each element are given in Table 1. The structural parameters are all interval variables and $\overline{E} = 2.058 \times 10^5$ (MPa), $\overline{\rho} = 7.65 \times 10^3$ (kg/m$^3$).

In order to investigate the effect of the interval variables $E$, $\rho$, $A$ and $L$ on the structural dynamic characteristics, different combinations for the values of interval ratios $\Delta E_f$, $\Delta\rho_f$, $\Delta A_f$ and $\Delta L_f$ are examined. The computational results for the natural frequencies and modeshapes are given in Table 2 and Table 3, respectively.

Table 1. The mean value of cross-sectional area of each element.

| Element | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ | $A_7$ | $A_8$ | $A_9$ | $A_{10}$ | $A_{11}$ | $A_{12}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Mean value ($\times 10^{-4}$ m$^2$) | 3 | 4 | 6 | 2 | 3 | 3 | 6 | 2 | 3 | 4 | 6 | 2 |

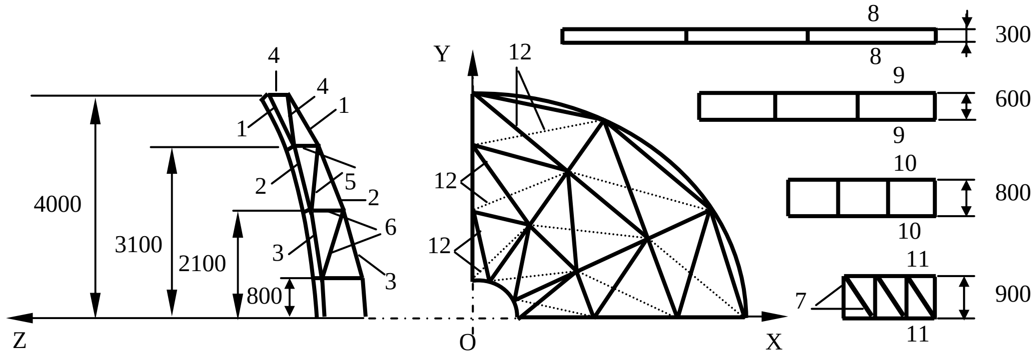


*Figure 1– Quarter of 8-meter caliber antenna (unit: mm)*

Table 2. Computational results for the natural frequencies.

| Model | $\omega_1^l$ | $\overline{\omega_1}$ | $\omega_1^u$ |
|---|---|---|---|
| Deterministic model $\Delta E_f = \Delta\rho_f = \Delta A_f = \Delta L_f = 0$ | 22.818 | 22.818 | 22.818 |
| $\Delta E_f = 0.1$ $\Delta\rho_f = \Delta A_f = \Delta L_f = 0$ | 21.645 | 22.788 | 23.931 |
| $\Delta\rho_f = 0.1$ $\Delta E_f = \Delta A_f = \Delta L_f = 0$ | 21.754 | 22.904 | 24.052 |
| $\Delta A_f = 0.1$ $\Delta E_f = \Delta\rho_f = \Delta L_f = 0$ | 22.818 | 22.818 | 22.818 |
| $\Delta L_f = 0.1$ $\Delta E_f = \Delta\rho_f = \Delta A_f = 0$ | 20.741 | 23.047 | 25.353 |
| $\Delta E_f = \Delta\rho_f = \Delta A_f = \Delta L_f = 0.1$ | 18.763 | 23.396 | 28.029 |
| $\Delta E_f = \Delta\rho_f = \Delta A_f = \Delta L_f = 0.2$ | 15.525 | 25.228 | 34.932 |

Table 3. Computational results for the modeshapes.

| Model | $\phi_{11}^l$ | $\overline{\phi_{11}}$ | $\phi_{11}^u$ |
|---|---|---|---|
| Deterministic model $\Delta E_f = \Delta\rho_f = \Delta A_f = \Delta L_f = 0$ | 2.1190 | 2.1190 | 2.1190 |
| $\Delta E_f = 0.1$ $\Delta\rho_f = \Delta A_f = \Delta L_f = 0$ | 2.1190 | 2.1190 | 2.1190 |
| $\Delta\rho_f = 0.1$ $\Delta E_f = \Delta A_f = \Delta L_f = 0$ | 2.0202 | 2.1269 | 2.2336 |
| $\Delta A_f = 0.1$ $\Delta E_f = \Delta\rho_f = \Delta L_f = 0$ | 2.0202 | 2.1269 | 2.2336 |
| $\Delta L_f = 0.1$ $\Delta E_f = \Delta\rho_f = \Delta A_f = 0$ | 2.0202 | 2.1269 | 2.2336 |
| $\Delta E_f = \Delta\rho_f = \Delta A_f = \Delta L_f = 0.1$ | 1.8367 | 2.1592 | 2.4817 |
| $\Delta E_f = \Delta\rho_f = \Delta A_f = \Delta L_f = 0.2$ | 1.6119 | 2.2866 | 2.9613 |

From Table 2 it can be observed that the effect of any uncertainty in the Young's modulus, density and length on the variability in the natural frequencies are different. Uncertainty in the length of a bar produces the greatest effect on the variability in the natural frequencies, but any variability in its cross-sectional area does not affect the natural frequencies. From Table 3 it can be observed that any uncertainty in the density, cross-sectional area and length of a bar produces the same effect on the change of the modeshapes. Variability in the Young's modulus does not have any effect on the modeshapes. Results show that when all the structural parameters have uncertainty in their values, the variability in the free vibrational characteristics of the structure is considerably greater than when only one of the parameters possesses uncertainty.

## CONCLUSIONS

In this paper, the effect of uncertainty in the material parameters and dimensions of the bars in a truss structure on the variability in the free vibrational characteristics of the structure is presented using a new technique called the interval factor method. With this method, the lower bound, upper bound and mean values of the natural frequencies and modeshapes were obtained.

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