ICSV13 - Vienna
The Thirteenth International Congress on Sound and Vibration
Vienna, Austria, July 2-6, 2006

A PERTURBATION METHOD FOR THE ACCURATE ESTIMATION OF THE VIBRATION SPECTRUM FOR THE TIMOSHENKO BEAM

Matthew P. Coleman

Department of Mathematics and Computer Science
Fairfield University
Fairfield, Connecticut 06824-5195, USA
mcoleman@mail.fairfield.edu

Abstract

In the vibration control of flexible structures, boundary feedback schemes are employed in order to damp vibrations and achieve stabilization. A knowledge of the system's vibration spectrum is crucial to this process. We have developed a perturbation approach that, when coupled with various asymptotic methods, yields highly accurate estimates for the vibration spectra of Euler–Bernoulli beam and Kirchhoff thin plate problems. Here, that method is extended to a stand-alone method, applicable to the Timoshenko beam equations. Of course, many such problems are solvable using commercially available software packages these days. The disadvantage here is that, as they generally employ FEM or similar numerical methods, these packages do not offer any of the analytical or physical insights that are provided by asymptotic methods. Thus, for example, our method allows us to see the similarities and differences between the Euler–Bernoulli, Rayleigh and Timoshenko beams. However, by their nature, asymptotic methods are least reliable at the low end of the spectrum where the most "important" frequencies – i.e., those corresponding to the greatest vibration energies – occur. This is especially true of Timoshenko beam problems. Here, then, the perturbations allow us vastly to improve these low-end estimates, to the point that excellent agreement with numerical results is obtained in every case we have tried.

INTRODUCTION

The Timoshenko beam equations constitute a model for a *thick* beam, as they incorporate the effects of rotary inertia and shear deformation. As such, the model is much more complicated than the Euler–Bernoulli beam, which leads to the subsequent difficulties in the analytic computation of its vibration spectrum. We extend to this problem a perturbation method that has been successful in the accurate computation of all ranges of the vibration spectrum of the Euler–Bernoulli beam [1], the slewing beam [2] and certain Kirchhoff thin plate problem [3]. Here, it is seen to be a stand-alone method that includes, as its first approximation, the asymptotic estimation of the spectrum which results from the application of the Wave Propagation Method of Keller and Rubinow ([4]) to the problem.

THE PROBLEM

We consider a Timoshenko beam of length L, with both ends strongly clamped. Letting $W(x,t)$ be the lateral displacement at point x at time t, and $\Phi(x,t)$ the bending angle at point x at time t, the Timoshenko beam is described by the following equations ([4]):

$$\rho W_{tt} - KW_{xx} + K\Phi_x = 0, \tag{1}$$

$$I_\rho \Phi_{tt} - EI\Phi_{xx} + K[\Phi - W_x] = 0, \qquad 0 < x < L. \tag{2}$$

Here, EI is the constant flexural rigidity, ρ the constant mass density, K the constant shear stiffness of a (uniform) cross section, I_ρ the constant rotary inertia, and $W_x = \frac{\partial W}{\partial x}$, etc. The boundary conditions are

$$W(0,t) = \Phi(0,t) = W(L,t) = \Phi(L,t) = 0, \qquad t > 0. \tag{3}$$

Now, we wish to find the eigenfrequencies λ; thus, we let

$$W(x,t) = w(x)e^{-i\lambda^2 a^2 t}, \quad \Phi(x,t) = \phi(x)e^{-i\lambda^2 a^2 t},$$

where $a^4 = \frac{EI}{\rho}$. Then, inserting (3) into (1) and (2) and eliminating ϕ, we arrive at

$$w^{(4)}(x) + 2r^2\lambda^4 w''(x) + (s^4 a^4 \lambda^8 - \lambda^4)w(x) = 0, \qquad 0 < x < L, \tag{4}$$

where $r^2 = \frac{1}{2}(\frac{EI}{K} + \frac{I_\rho}{\rho})$ and $s^4 = \frac{I_\rho}{K}$. Similarly, one may show, as in [4], that the separated boundary conditions are

$$\begin{gathered} w(0) = w(L) = 0, \\ \frac{EI}{K}w'''(0) + \left[1 + \frac{\rho a^4 \lambda^4 EI}{K^2}\right] w'(0) = \frac{EI}{K}w'''(L) + \left[1 + \frac{\rho a^4 \lambda^4 EI}{K^2}\right] w'(L) = 0. \end{gathered} \tag{5}$$

APPLICATION OF THE PERTURBATION METHOD

Letting $w = e^{\alpha x}$ in (4), we find that

$$\alpha = \pm i\lambda^2\phi_1, \quad \pm i\lambda^2\phi_2,$$

where

$$\phi_1 = \sqrt{r^2 - \sqrt{r^4 - s^4a^4 + \frac{1}{\lambda^4}}}, \quad \phi_2 = \sqrt{r^2 + \sqrt{r^4 - s^4a^4 + \frac{1}{\lambda^4}}}\ .$$

Applying the boundary conditions (5) to the general solution

$$w = Ae^{i\lambda^2\phi_1 x} + Be^{i\lambda^2\phi_2 x} + Ce^{-i\lambda^2\phi_1 x} + De^{-i\lambda^2\phi_2 x}$$

leads to

$$\begin{aligned}
&A + B + C + D = 0,\\
&f_1A + f_2B - f_1C - f_2D = 0,\\
&Ae^{i\lambda^2\phi_1 L} + Be^{i\lambda^2\phi_2 L} + Ce^{-i\lambda^2\phi_1 L} + De^{-i\lambda^2\phi_2 L} = 0,\\
&Af_1e^{i\lambda^2\phi_1 L} + Bf_2e^{i\lambda^2\phi_2 L} - Cf_1e^{-i\lambda^2\phi_1 L} - Df_2e^{-i\lambda^2\phi_2 L} = 0.
\end{aligned}$$

Here,

$$f_i = -KEI\phi_i^3 + \rho a^4 EI\phi_i + \frac{K^2}{\lambda^4}\phi_i, \qquad i = 1, 2.$$

Our eigenfrequencies, then, are determined by the condition that

$$\begin{aligned}
0 &= \begin{vmatrix} 1 & 1 & 1 & 1\\ f_1 & f_2 & -f_1 & -f_2\\ e^{i\lambda^2\phi_1 L} & e^{i\lambda^2\phi_2 L} & e^{-i\lambda^2\phi_1 L} & e^{-i\lambda^2\phi_2 L}\\ f_1e^{i\lambda^2\phi_1 L} & f_2e^{i\lambda^2\phi_2 L} & -f_1e^{-i\lambda^2\phi_1 L} & -f_2e^{-i\lambda^2\phi_2 L} \end{vmatrix} = \det M\\
&= (f_1 - f_2)^2 e^{i\lambda^2 L(\phi_1+\phi_2)} - (f_1 + f_2)^2 e^{i\lambda^2 L(\phi_1-\phi_2)} - (f_1 + f_2)^2 e^{i\lambda^2 L(\phi_2-\phi_1)}\\
&\quad + (f_1 - f_2)^2 e^{-i\lambda^2 L(\phi_1+\phi_2)} + 8f_1f_2. \qquad (6)
\end{aligned}$$

Now, at this point, standard asymptotic methods would neglect all terms of $O(\frac{1}{\lambda^4})$, as in [4]. Guided by our previous work ([1, 2, 3]), we, instead, replace $\frac{1}{\lambda^4}$ with ϵ, with the idea of using $\epsilon = \frac{1}{\lambda^4}$ in the implementation of our asymptotic method.

We have, then,

$$\begin{aligned}
\phi_1(\epsilon) &= \sqrt{r^2 - \sqrt{r^4 - s^4a^4 + \epsilon}} =\\
&= \sqrt{r^2 - \sqrt{r^4 - s^4a^4}} - \frac{1}{4\sqrt{r^2 - \sqrt{r^4 - s^4a^4}}\sqrt{r^4 - s^4a^4}}\epsilon + O(\epsilon^2)\\
\phi_2(\epsilon) &= \sqrt{r^2 + \sqrt{r^4 - s^4a^4}} + \frac{1}{4\sqrt{r^2 - \sqrt{r^4 - s^4a^4}}\sqrt{r^4 - s^4a^4}}\epsilon + O(\epsilon^2)
\end{aligned}$$

which, upon simplification, become, for $\frac{EI}{K} > \frac{I_\rho}{\rho}$,

$$\phi_1(\epsilon) = b + \frac{1}{2b(b^2 - a^2)}\epsilon + O(\epsilon^2),$$

$$\phi_2(\epsilon) = a + \frac{1}{2a(a^2 - b^2)}\epsilon + O(\epsilon^2),$$

where $a = \sqrt{\frac{EI}{K}}$ and $b = \sqrt{\frac{I_\rho}{\rho}}$ (and where we have abused notation, as this a is not the same as the a for which $a^4 = \frac{EI}{\rho}$). Now, for $\frac{I_\rho}{\rho} > \frac{EI}{K}$, we have

$$\phi_1(\epsilon) = a + \frac{1}{2a(a^2-b^2)}\epsilon + O(\epsilon^2),$$
$$\phi_2(\epsilon) = b + \frac{1}{2b(b^2-a^2)}\epsilon + O(\epsilon^2).$$

Thus, WLOG, we may assume that $\frac{EI}{K} > \frac{I_\rho}{\rho}$. Then, after much simplification, we find that

$$f_1(\epsilon) = K^2\left[a^2b(a^2-b^2) + \frac{a^4-5a^2b^2+2b^4}{2b(b^2-a^2)}\epsilon\right] + O(\epsilon^2),$$
$$f_2(\epsilon) = K^2\frac{ab^2}{b^2-a^2}\epsilon + O(\epsilon^2).$$

Further, we expand the eigenvalues as

$$\lambda = \lambda_0 + \lambda_1\epsilon + O(\epsilon^2)$$

and, we have

$$e^{\pm i\lambda^2\phi_1 L} = e^{\pm i\lambda_0^2 bL}\left\{1 \pm \epsilon i L\left[2\lambda_0\lambda_1 b + \frac{\lambda_0^2}{2b(b^2-a^2)}\right]\right\} + O(\epsilon^2),$$
$$e^{\pm i\lambda^2\phi_2 L} = e^{\pm i\lambda_0^2 aL}\left\{1 \pm \epsilon i L\left[2\lambda_0\lambda_1 a + \frac{\lambda_0^2}{2b(b^2-a^2)}\right]\right\} + O(\epsilon^2).$$

Thus, after much simplification, we may write the determinant equation (6) as

$$\begin{aligned}\det M(\epsilon) = C_1^2[&e^{i\lambda_0^2L(a+b)} - e^{i\lambda_0^2L(b-a)} - e^{i\lambda_0^2L(a-b)} + e^{-i\lambda_0^2L(a+b)}]\\
&+\epsilon\Big\{\Big[2C_1(C_2-C_3) + iLC_1^2\cdot 2\lambda_0\lambda_1(a+b) - iLC_1^2\frac{\lambda_0^2}{2ab(a+b)}\Big]e^{i\lambda_0^2L(a+b)}\\
&+\Big[-2C_1(C_2+C_3) - iLC_1^2\cdot 2\lambda_0\lambda_1(b-a) - iLC_1^2\frac{\lambda_0^2}{2ab(b-a)}\Big]e^{i\lambda_0^2L(b-a)}\\
&+\Big[-2C_1(C_2+C_3) - iLC_1^2\cdot 2\lambda_0\lambda_1(a-b) - iLC_1^2\frac{\lambda_0^2}{2ab(a-b)}\Big]e^{i\lambda_0^2L(a-b)}\\
&+\Big[2C_1(C_2-C_3) - iLC_1^2 2\lambda_0\lambda_1(a+b) + iLC_1^2\frac{\lambda_0^2}{2ab(a+b)}\Big]e^{-i\lambda_0^2L(a+b)}\\
&-8a^3b^3\Big\} + O(\epsilon^2) = 0.\end{aligned} \tag{7}$$

$$C_1 = a^2b(a^2-b^2),$$
$$C_2 = \frac{a^4-5a^2b^2+2b^4}{2b(b^2-a^2)},$$
$$C_3 = \frac{ab^2}{b^2-a^2}.$$

Now, the "first approximation" $\det M(0) = 0$, which is the 0^{th}-order approximation resulting from setting the coefficient of ϵ^0 to zero, can be shown to be equivalent to the application of the Wave Propagation Method to the problem. In order to improve our approximation, we next set the coefficient of ϵ^1 in (7) equal to zero, and solve for λ_1 in terms of λ_0.

At this point, the question becomes what to use for ϵ. We choose to call $\lambda_0 = \lambda^{(0)}$ and set

$$\epsilon = \epsilon_0 = \frac{1}{(\lambda^{(0)})^4},$$

from which we have our improved estimate

$$\lambda^{(1)} = \lambda_0 + \epsilon_0 \lambda_1.$$

The process may be continued recursively and indefinitely:

$$\epsilon_i = \frac{1}{(\lambda^{(i)})^4} \quad \text{and} \quad \lambda^{(i+1)} = \lambda_0 + \epsilon_i \lambda_1.$$

RESULTS AND COMPARISONS

We present two examples. In each case, we have computed numerical results using the Legendre-tau spectral method. For Example 1, we use $\rho = 1$, $I_\rho = 2$, $I = 3$, $E = 2.5$, $K = 1.5$ and $L = 0.1$. The results can be seen in Table 1. The numerical results (LT) here have converged to at least 8 decimal places. We have listed the first six frequencies, after which the numerical results match the 0th-order approximations to 6 decimal places.

We have used for our Example 2 the built-in box beam found in [6]. Here, we have $\rho = .0038$, $I_\rho = .00455$, $I = 2830$, $E = 10,000$, $K = 940,000$ and $L = 19$. In this case, the numerical results have converged to four decimal places. We have found that it is much more difficult, here, to get agreement between the two methods. However, we see that the perturbation results still approach the numerical results. We list the 25th frequency as, there, we have exact agreement after four iterations, and we have found close agreement thereafter.

Table 1: Eigenfrequencies from Example 1.

	1st	2nd	3rd	4th	5th	6th
LT	40.5121	64.2007	85.8129	123.051	129.223	161.084
$\lambda^{(0)}$	40.5140	64.2088	85.8148	123.054	129.222	161.084
$\lambda^{(1)}$	40.5115	64.1974	85.8123	123.051	129.223	
$\lambda^{(2)}$	40.5124	64.2027	85.8131			
$\lambda^{(3)}$	40.5122	64.1992	85.8129			
$\lambda^{(4)}$	40.5122	64.2011				
$\lambda^{(5)}$		64.2011				

Table 2: Eigenfrequencies from Example 2.

	1st	2nd	3rd	–	25th
LT	2331.	5002.	7770.		54160.
$\lambda^{(0)}$	2815.	5456.	8099.		54050.
$\lambda^{(1)}$	2082.	4822.	7665.		54220.
$\lambda^{(2)}$	2472.	5119.	7852.		54140
$\lambda^{(3)}$	2240.	4930.	7717.		54170.
$\lambda^{(4)}$	2391.	5044.	7778.		54160.
$\lambda^{(5)}$	2293.	5021.	7778.		
$\lambda^{(6)}$	2302.	5021.			
$\lambda^{(7)}$	2302.				

IN CLOSING

The perturbation method exhibited here gives excellent agreement with numerical results in many cases. In those cases where the agreement is not as good, greater accuracy may be achieved by including terms corresponding to higher powers of ϵ in the expansion of λ.

References

[1] G. Chen, M. P. Coleman, "Improving low order eigenfrequency estimates derived from the wave propagation method for an Euler-Bernoulli beam", Journal of Sound and Vibration, **204 (4)**, 696-704 (1997)

[2] M. P. Coleman, "Vibration eigenfrequency analysis of a single-link flexible manipulator", Journal of Sound and Vibration, **212 (1)**, 109-120 (1998)

[3] M. P. Coleman, L. A. McSweeney, "A perturbation approach for the eigenfrequency analysis of separable Kirchhoff thin plate problems", accepted by Journal of Sound and Vibration

[4] J. B. Keller, S. I. Rubinow, "Asymptotic solution of eigenvalue problems", Annals of Physics, **9**, 24-75 (1960)

[5] M. P. Coleman, H. K. Wang, "Analysis of vibration spectrum of a Timoshenko beam with boundary damping by the wave method", Wave Motion, **17 (3)**, 223-239 (1993)

[6] R. W. Traill-Nash, A. R. Collar, "The effect of shear flexibility and rotary inertia on the bending vibration of beams", Quarterly Journal of Mechanics and Applied Mathematics, **6**, 186-222 (1953)

[7] J. U. Kim, Y. Renardy, "Boundary control of the Timoshenko beam", SIAM Journal of Optimization and Control, **25**, 1417-1429 (1987).