ICSV13 - Vienna
The Thirteenth International Congress
on Sound and Vibration
Vienna, Austria, July 2-6, 2006

A SYSTEMATIC APPROACH FOR LINEAR ACOUSTIC MODELLING OF COMPLEX SILENCER SYSTEMS

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Abstract

A mathematical formulation for linear plane-wave acoustic modelling of complex silencer systems containing multi-port elements has been developed. The prevalence of such multi-port elements in automotive silencers — perforated pipe elements and crossovers being typical examples — dictates the need for a simulation tool that facilitates their integration into an acoustic network comprising lumped and distributed elements.

A fundamental aspect of the presented approach evolves from the individual mathematical characterization of lumped and distributed elements: lumped elements are governed by algebraic equations; distributed elements by differential equations. The resulting matrices thus have distinctive forms: matrices similar to the so-called impedance form are used for lumped elements; scattering-matrices for distributed elements. This distinction leads consequently to an object-oriented software design with a set of classes reflecting the unique properties associated with each of the acoustic element classifications.

A procedure for the automated assembly of the matrix system for the complete network is introduced, in which the problem is intrinsically reduced to the smallest possible number of unknowns. The solution delivers a vector of travelling pressure waves entering the distributed elements at their connections. By calculating the pressure waves leaving the distributed elements via the scattering matrices, the sound field for the entire system can be easily obtained. If one is only interested in the transmission behaviour of a complex system, the appropriate scattering matrix can be obtained directly from matrix condensation by eliminating all inner degrees of freedom. Since source terms can be added to both lumped and distributed elements, active systems can also be modelled using the presented approach.

INTRODUCTION

Acoustic 2-port (4-pole) methods represent a well-established elementary technique for simulating wave propagation phenomena in automotive exhaust systems. Despite differences in the choice of state variables, a common aspect of the various formulations [1, 2] is the use of closed-form, frequency-domain solutions of single acoustic transfer elements. These methods lend themselves to systems with a single direction of sound power flux, such as cascaded elements having side branches with closed ends. For systems with a multiple sound power flux paths, such as commonly encountered in duct networks, alternative formulations have been introduced [3, 4].

These formulations are restricted to 2-port wave guides, which constitutes a limitation for automotive exhaust applications, as it precludes distributed models of perforated pipe elements. A "tri-flow" muffler, for example, comprises three perforated pipes within a casing and requires an 8-port wave guide element for its plane wave acoustic description. For such elements, a discrete modelling approach provides a means to overcome this limitation [5]. In this approach, the perforated pipes are represented as a network of 2-port elements that can be integrated into the formulation of Glav and Åbom [4]. For highly perforated pipes (*e.g.,* liners of absorption mufflers), numerical problems may however be associated with the fine discretization. The novel modelling strategy presented here facilitates the direct integration of distributed models of multi-port wave guides into a numerical model of a complex duct network.

MODELLING OF SINGLE ELEMENTS

Acoustic elements can be classified into distributed and lumped elements (see Fig. 1). Distributed elements are characterized by their length, which is significantly larger than their transverse dimensions. At sufficiently low frequencies, the cross-sectional shape is irrelevant and the wave propagation can be treated as one-dimensional along the length. Distributed elements are also termed "wave guides" (notated by the index g) in the following.

In contrast, lumped elements have no particular wave propagation length, but act as connectors to describe the transition of the sound power flux between wave guides. The lumped elements are thus termed "nodal elements" (notated by the index n) in the following. The cross-sectional area of lumped elements is of importance, especially when describing discontinuities between the connected wave guides.

Distributed Elements — Wave Guides

The sound wave propagation within one-dimensional passive wave guides is described by a set of homogenous partial differential equations derived from mass, momentum and energy conservation (for example, see Munjal [1]). Dokumaci [6, 7] suggested that a state space form is preferable to deriving a wave equation for the sound pressure. For the mathematical formulation, an extended state vector

$$\mathbf{s}(x,t) = \{p_1(x,t), \rho_1(x,t), u_1(x,t), \ldots, p_n(x,t), \rho_n(x,t), u_n(x,t)\}^\top \ , \tag{1}$$

can be used that comprises the quantities describing the oscillation state of the fluid – sound pressure p, density fluctuation ρ and sound velocity u. The state vector is a function of the element coordinate x and of time t. The number of geometric ports at each end of a wave guide is denoted by the index n (*i.e.,* a wave guide has a total of $2n$ geometric ports). The

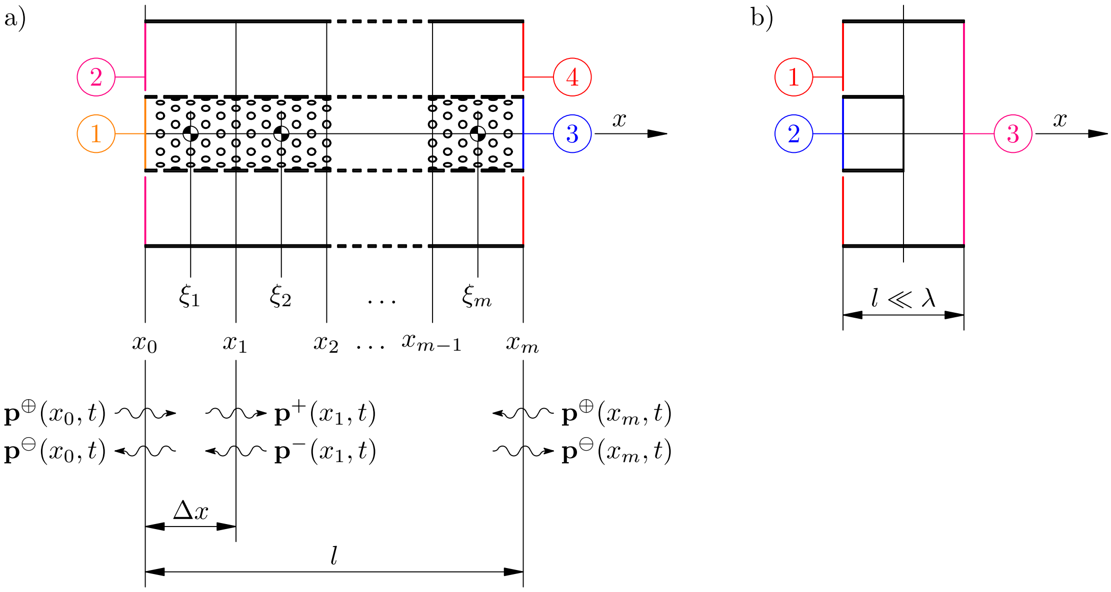

Figure 1: *Examples of distributed and lumped multi-port elements. a) perforated 2-duct element; b) fork element. Numbers in circles refer to the geometric ports.*

linearized governing equations can be written in state space form as follows:

$$\frac{\partial}{\partial x}\mathbf{s}(x,t) - \mathbf{B}_1(x)\frac{\partial}{\partial t}\mathbf{s}(x,t) - \mathbf{B}_2(x)\mathbf{s}(x,t) = \mathbf{0}\ . \tag{2}$$

Using the harmonic transformation $\mathbf{s}(x,t) = \underline{\mathbf{s}}(x,\omega)\,\mathrm{e}^{\mathrm{j}\omega t}$ yields the governing equation for complex-valued state vector

$$\frac{\partial}{\partial x}\underline{\mathbf{s}}(x,\omega) = \mathbf{B}(x,\omega)\underline{\mathbf{s}}(x,\omega) \tag{3}$$

with the complex-valued coefficient matrix $\mathbf{B}(x,\omega) = [\,\mathrm{j}\omega\mathbf{B}_1(x) + \mathbf{B}_2(x)\,]$. For isentropic wave propagation, the density fluctuations ρ are proportional to the sound pressures p. To avoid degeneracy of the coefficient matrix $\mathbf{B}(x,\omega)$, the dependence of the complex-valued state vector on density fluctuations is removed, but can be subsequently recovered via $\rho = p/c_0^2$.

From the acoustic state equation (3), the so-called matrizant approach [8] can be used to derive the transfer matrix for a distributed element:

$$\underline{\mathbf{s}}(x_m,\omega) = [\mathbf{B}(x,\omega)]_{x_0}^{x_m}\,\underline{\mathbf{s}}(x_0,\omega)\ , \tag{4}$$

in which $[\mathbf{B}(x,\omega)]_{x_0}^{x_m}$ denotes the matrizant of the coefficient matrix $\mathbf{B}(x,\omega)$ over the interval (x_0,x_m). For the numerical evaluation of the matrizant, the wave guide is divided into m equal lengths $\Delta x = l/m$ (see Fig. 1). The complete matrizant is obtained by successive multiplication of the incremental matrizants $[\mathbf{B}(x,\omega)]_{x_{i-1}}^{x_i}$ for $i = 1\ldots m$. For sufficiently small Δx increments, the x-dependant matrix coefficients can be replaced by their mid-point evaluations at $\xi_i = (x_{i-1}+x_i)/2$.

$$[\mathbf{B}(x,\omega)]_{x_0}^{x_m} \approx [\mathbf{B}(\xi_m,\omega)]_{x_{m-1}}^{x_m}\cdots[\mathbf{B}(\xi_2,\omega)]_{x_1}^{x_2}\,[\mathbf{B}(\xi_1,\omega)]_{x_0}^{x_1} \tag{5}$$

The matrizant evaluation thus reduces to a matrix product of exponential matrix functions. Introducing the naming convention $\mathbf{B}_i = \mathbf{B}(\xi_i, \omega)$ allows the incremental matrizant to be expressed as follows:

$$[\mathbf{B}(\xi_i, \omega)]_{x_{i-1}}^{x_i} = \mathrm{e}^{\mathbf{B}_i} = \mathbf{\Psi}_i(\omega)\,\mathrm{diag}\left(\mathrm{e}^{\boldsymbol{\beta}_i(\omega)\Delta x}\right)\mathbf{\Psi}_i^{-1}(\omega)\,, \tag{6}$$

where $\mathbf{\Psi}_i(\omega)$ denotes the matrix of left eigenvectors and $\boldsymbol{\beta}_i(\omega)$ the vector of eigenvalues of the coefficient matrix $\mathbf{B}_i$. The diag operator yields the diagonal matrix corresponding to its vector argument. Substituting (6) into (5) leads to the expression

$$\begin{aligned}[\mathbf{B}(x, \omega)]_{x_0}^{x_m} &\approx \prod_{i=m}^{1} \mathbf{\Psi}_i(\omega)\,\mathrm{diag}\left(\mathrm{e}^{\boldsymbol{\beta}_i(\omega)\Delta x}\right)\mathbf{\Psi}_i^{-1}(\omega) \\ &\approx \mathbf{\Psi}_m(\omega)\,\mathrm{diag}\left(\mathrm{e}^{\Delta x \sum\limits_{i=1}^{m} \boldsymbol{\beta}_i(\omega)}\right)\mathbf{\Psi}_1^{-1}(\omega)\,,\end{aligned} \tag{7}$$

since the matrix product of successive eigenvector matrices $\mathbf{\Psi}_i^{-1}(\omega)\mathbf{\Psi}_{i-1}(\omega) \approx \mathbf{I}$ holds for small Δx, provided that the eigenvalues and eigenvectors are ordered the same for all incremental matrizants. From the transfer matrix (7), it can be seen that a new complex-valued state vector $\underline{\mathbf{s}}^{\pm}(\omega)$ can be introduced. Substituting (7) into (4), and multiplying by the inverse of the eigenvector matrix $\mathbf{\Psi}_m^{-1}(\omega)$ from the left side, results in the transformation between the state vectors:

$$\underline{\mathbf{s}}^{\pm}(x_0, \omega) = \mathbf{\Psi}_1^{-1}(\omega)\underline{\mathbf{s}}(x_0, \omega) \quad \text{and} \quad \underline{\mathbf{s}}^{\pm}(x_m, \omega) = \mathbf{\Psi}_m^{-1}(\omega)\underline{\mathbf{s}}(x_m, \omega)\,. \tag{8}$$

Setting the first component of each eigenvector $\boldsymbol{\psi}_j(\omega)$ to unity allows the components of the state vector $\underline{\mathbf{s}}^{\pm}(x, \omega) = \{\underline{\mathbf{p}}^{+}(x, \omega), \underline{\mathbf{p}}^{-}(x, \omega)\}^{\top}$ to be interpreted as the progressive $\underline{p}_j^{+}(x, \omega)$ and reflected $\underline{p}_j^{-}(x, \omega)$ pressure waves. Since the wave propagation constants are governed by the eigenvalues, the distinction between progressive and reflected waves arises from the sign of the imaginary components $\mathrm{Im}(\beta_j)$. By partitioning the set of eigenvalues into $\{\beta_j^{+}(\omega)\}$ and $\{\beta_j^{-}(\omega)\}$, which correspond to $\{\beta_j(\omega)|\mathrm{Im}(\beta_j(\omega)) < 0\}$ and $\{\beta_j(\omega)|\mathrm{Im}(\beta_j(\omega)) > 0\}$, respectively, one obtains the sets of eigenvectors $\{\boldsymbol{\psi}_j^{+}(\omega)\}$ and $\{\boldsymbol{\psi}_j^{-}(\omega)\}$.

Using the following notation for the exponential matrices of the sets of eigenvalues,

$$\mathbf{G}^{+}(\omega) = \mathrm{diag}\left(\mathrm{e}^{\Delta x \sum\limits_{i=1}^{m} \{\beta_j^{+}(\omega)\}_i^{\top}}\right) \quad \text{and} \quad \mathbf{G}^{-}(\omega) = \mathrm{diag}\left(\mathrm{e}^{\Delta x \sum\limits_{i=1}^{m} \{\beta_j^{-}(\omega)\}_i^{\top}}\right)\,, \tag{9}$$

results in the scattering matrix of a wave guide between points 0 and m :

$$\begin{Bmatrix} \underline{\mathbf{p}}^{+}(x_m, \omega) \\ \underline{\mathbf{p}}^{-}(x_m, \omega) \end{Bmatrix} = \begin{bmatrix} \mathbf{G}^{+}(\omega) & \mathbf{0} \\ \mathbf{0} & \mathbf{G}^{-}(\omega) \end{bmatrix} \begin{Bmatrix} \underline{\mathbf{p}}^{+}(x_0, \omega) \\ \underline{\mathbf{p}}^{-}(x_0, \omega) \end{Bmatrix}\,. \tag{10}$$

This "classical" definition of the scattering matrix is, however, inappropriate for modelling a duct network, since it constitutes a preferential bias for ports at point 0 versus ports at point m of a wave guide. Instead, a modified sign convention is introduced to distinguish between pressure waves entering and leaving the wave guide: incoming pressure waves $\underline{\mathbf{p}}_g^{\oplus}(\omega) = \{\underline{\mathbf{p}}^{+}(x_0, \omega), \underline{\mathbf{p}}^{-}(x_m, \omega)\}^{\top}$ and outgoing pressure waves $\underline{\mathbf{p}}_g^{\ominus}(\omega) = \{\underline{\mathbf{p}}^{-}(x_0, \omega), \underline{\mathbf{p}}^{+}(x_m, \omega)\}^{\top}$.

When active wave guides are considered, the source terms that should appear in the governing equations can be included using a simplified approach. Based on the independence of passive and active components, the source vector $\underline{\mathbf{q}}_g^{\ominus}(\omega) = \{\underline{\mathbf{q}}^{\ominus}(x_0,\omega), \underline{\mathbf{q}}^{\ominus}(x_m,\omega)\}^{\top}$ is added to the vector of outgoing pressure waves $\underline{\mathbf{p}}_g^{\ominus}(\omega)$. Using the above definitions of the state vectors $\underline{\mathbf{p}}_g^{\oplus}(\omega)$ and $\underline{\mathbf{p}}_g^{\ominus}(\omega)$, a scattering matrix for wave guides

$$\mathbf{G}(\omega) = \begin{bmatrix} \mathbf{0} & (\mathbf{G}^{-}(\omega))^{-1} \\ \mathbf{G}^{+}(\omega) & \mathbf{0} \end{bmatrix} \tag{11}$$

can be derived from (10) that yields the following relationship between the state vectors and the source vector:

$$\underline{\mathbf{p}}_g^{\ominus}(\omega) = \mathbf{G}(\omega)\underline{\mathbf{p}}_g^{\oplus}(\omega) + \underline{\mathbf{q}}_g^{\ominus}(\omega) \ . \tag{12}$$

The relationship between the state vector $\underline{\mathbf{s}}_g(\omega) = \{\underline{\mathbf{s}}(x_0,\omega), \underline{\mathbf{s}}(x_m,\omega)\}^{\top}$ and the incoming $\underline{\mathbf{p}}_g^{\oplus}(\omega)$ and outgoing $\underline{\mathbf{p}}_g^{\ominus}(\omega)$ pressure waves is defined by the transformation matrices

$$\mathbf{T}^{\oplus}(\omega) = \begin{bmatrix} \mathbf{\Psi}_1^{+}(\omega) & \mathbf{0} \\ \mathbf{0} & \mathbf{\Psi}_m^{-}(\omega) \end{bmatrix} \quad \text{and} \quad \mathbf{T}^{\ominus}(\omega) = \begin{bmatrix} \mathbf{\Psi}_1^{-}(\omega) & \mathbf{0} \\ \mathbf{0} & \mathbf{\Psi}_m^{+}(\omega) \end{bmatrix} , \tag{13}$$

which gives the relationship

$$\underline{\mathbf{s}}_g(\omega) = \mathbf{T}^{\oplus}(\omega)\underline{\mathbf{p}}_g^{\oplus}(\omega) + \mathbf{T}^{\ominus}(\omega)\underline{\mathbf{p}}_g^{\ominus}(\omega) \ . \tag{14}$$

For isentropic wave propagation, the aforementioned density fluctuation and sound pressure relationship can be reintroduced into the eigenvector matrices $\mathbf{\Psi}_1^{+}(\omega)$, $\mathbf{\Psi}_1^{-}(\omega)$, $\mathbf{\Psi}_m^{+}(\omega)$ and $\mathbf{\Psi}_m^{-}(\omega)$.

Substituting (12) into (14) leads to a final relationship between the state vector, the incoming pressure waves and the source vector:

$$\underline{\mathbf{s}}_g(\omega) = \mathbf{T}(\omega)\underline{\mathbf{p}}_g^{\oplus}(\omega) + \underline{\mathbf{q}}_g(\omega) \ , \tag{15}$$

in which a transformation matrix

$$\mathbf{T}(\omega) = \mathbf{T}^{\oplus}(\omega) + \mathbf{T}^{\ominus}(\omega)\mathbf{G}(\omega) \tag{16}$$

and a transformed source vector

$$\underline{\mathbf{q}}_g(\omega) = \mathbf{T}^{\ominus}(\omega)\underline{\mathbf{q}}_g^{\ominus}(\omega) \tag{17}$$

have been defined. The equations (12) and (15) constitute the basis for the class interface for distributed elements. The class interface delivers the scattering matrix $\mathbf{G}(\omega)$, the source vector $\underline{\mathbf{q}}_g^{\ominus}(\omega)$, the transformation matrix $\mathbf{T}(\omega)$ and the transformed source vector $\underline{\mathbf{q}}_g(\omega)$.

Lumped Elements — Nodal Elements

In contrast to distributed elements, lumped elements lack a spatial component. As a result of which, the governing equations are much simpler. In many cases, the control volume for the conservation equations may even degrade to a control area and the governing equations to a quasi-steady form. Lumped inertance and lumped compliance elements cannot however be treated as quasi-steady. Using a state vector $\mathbf{s}_n(t) = \{p(t), \rho(t), u(t), \ldots, p_n(t), \rho_n(t), u_n(t)\}^\top$ that includes the acoustic state variables of all n geometric ports of a lumped element, and considering inertia terms, the linearized governing equations can be written in the form

$$\mathbf{N}_1 \frac{\partial}{\partial t}\mathbf{s}_n(t) + \mathbf{N}_2 \mathbf{s}_n(t) = \mathbf{q}_n(t) \ , \tag{18}$$

in which $\mathbf{q}_n(t)$ is the source vector.

Using the frequency transformations $\mathbf{s}_n(t) = \underline{\mathbf{s}}_n(\omega)\,\mathrm{e}^{\mathrm{j}\omega t}$ and $\mathbf{q}_n(t) = \underline{\mathbf{q}}_n(\omega)\,\mathrm{e}^{\mathrm{j}\omega t}$, allows a complex-valued coefficient matrix $\mathbf{N}(\omega) = [\mathrm{j}\omega\mathbf{N}_1 + \mathbf{N}_2]$ to be introduced that leads to the final form for nodal elements:

$$\mathbf{N}(\omega)\underline{\mathbf{s}}_n(\omega) = \underline{\mathbf{q}}_n(\omega) \tag{19}$$

The class interface for nodal elements delivers the nodal matrix $\mathbf{N}(\omega)$ and the source vector $\underline{\mathbf{q}}_n(\omega)$ accordingly.

MODELLING THE COMPLETE NETWORK

Modelling the network implies assembling the matrix equations of all wave guides and nodal elements. A new state vector for the complete network is introduced $\underline{\mathbf{s}}(\omega) = \{\underline{\mathbf{s}}_{g,1}(\omega), \ldots, \underline{\mathbf{s}}_{g,n}(\omega)\}^\top$ that is simply a column vector containing the state vectors $\underline{\mathbf{s}}_{g,i}(\omega)$ of all $i = 1 \ldots n$ wave guides. The vectors of incoming $\underline{\mathbf{p}}^\oplus(\omega)$ and outgoing $\underline{\mathbf{p}}^\ominus(\omega)$ pressure waves, and the wave guide source vector $\underline{\mathbf{q}}^\ominus(\omega)$ are defined similarly. After combining the scattering matrices into a single block diagonal matrix $\mathbf{G}_s(\omega) = \mathrm{diag}\,(\mathbf{G}_1(\omega), \ldots, \mathbf{G}_n(\omega))$, the relationship between incoming and outgoing pressure waves can be determined for the complete network system (notated by the index s):

$$\underline{\mathbf{p}}^\ominus(\omega) = \mathbf{G}_s(\omega)\underline{\mathbf{p}}^\oplus(\omega) + \underline{\mathbf{q}}^\ominus(\omega) \ . \tag{20}$$

Establishing a network transformation matrix $\mathbf{T}_s(\omega) = \mathrm{diag}\,(\mathbf{T}_1(\omega), \ldots, \mathbf{T}_n(\omega))$ and a transformed source vector $\underline{\mathbf{q}}_g(\omega) = \{\underline{\mathbf{q}}_{g,1}(\omega), \ldots, \underline{\mathbf{q}}_{g,n}(\omega)\}^\top$ allows the network state vector $\underline{\mathbf{s}}(\omega)$ to be expressed as a function of the vector of the incoming pressure waves:

$$\underline{\mathbf{s}}(\omega) = \mathbf{T}_s(\omega)\underline{\mathbf{p}}^\oplus(\omega) + \underline{\mathbf{q}}_g(\omega) \ . \tag{21}$$

To match the sort sequence of the state vectors for the wave guides $\underline{\mathbf{s}}_{g,i}(\omega)$ with that of the nodal elements $\underline{\mathbf{s}}_{n,j}(\omega)$, a projection or permutation matrix $\mathbf{\Pi}$ is introduced that reorders the vector components:

$$\{\underline{\mathbf{s}}_{n,1}(\omega), \ldots, \underline{\mathbf{s}}_{n,m}(\omega)\}^\top = \mathbf{\Pi}\ \{\underline{\mathbf{s}}_{g,1}(\omega), \ldots, \underline{\mathbf{s}}_{g,n}(\omega)\}^\top \ . \tag{22}$$

The projection matrix consists of 3×3 unity matrices $\mathbf{I}_3$, since the dynamic state at each port is described by the three quantities p, ρ and u. After introducing a block diagonal matrix $\mathbf{N}_s(\omega) = \mathrm{diag}\,(\mathbf{N}_1(\omega), \ldots, \mathbf{N}_m(\omega))$ for a network of $j = 1 \ldots m$ nodal elements and the corresponding source vector $\underline{\mathbf{q}}_n(\omega) = \{\underline{\mathbf{q}}_{n,1}(\omega), \ldots, \underline{\mathbf{q}}_{n,m}(\omega)\}^\top$, one can write a nodal equation for the complete network:

$$\mathbf{N}_s(\omega)\,\mathbf{\Pi}\,\underline{\mathbf{s}}(\omega) = \underline{\mathbf{q}}_n(\omega) \ . \tag{23}$$

Setup of the Network Equation

Substituting (21) into (23) defines the system matrix

$$\mathbf{S}(\omega) = \mathbf{N}_s(\omega)\,\mathbf{\Pi}\,\mathbf{T}_s(\omega) \tag{24}$$

and the source vector of the duct network

$$\underline{\mathbf{q}}(\omega) = \underline{\mathbf{q}}_n(\omega) - \mathbf{N}_s(\omega)\,\mathbf{\Pi}\,\underline{\mathbf{q}}_g(\omega)\ . \tag{25}$$

With these definitions, the matrix equation of the complete duct network that allows the determination of the vector of the incoming pressure waves for all wave guides can be written as follows

$$\mathbf{S}(\omega)\underline{\mathbf{p}}^{\oplus}(\omega) = \underline{\mathbf{q}}(\omega)\ . \tag{26}$$

It is important to observe that the total number of unknowns of a network system is determined solely by the wave guides.

Scattering and Transformation Matrices of a Complex Multi-Port Element

To obtain the scattering matrix $\mathbf{G}(\omega)$ of a complex element comprising several wave guides and nodal elements, virtual one-port nodes are attached to the ports corresponding to the n geometrical ports of the complex element. These ports are termed “boundary ports” (index b). In contrast, the “inner ports” (index i) are ports at which the wave guides of the complex element connect to each other. The network equation (26) can thus be partitioned into the following matrix equation

$$\begin{bmatrix}\mathbf{S}_{bb}(\omega) & \mathbf{S}_{bi}(\omega)\\ \mathbf{S}_{ib}(\omega) & \mathbf{S}_{ii}(\omega)\end{bmatrix}\begin{Bmatrix}\underline{\mathbf{p}}_b^{\oplus}(\omega)\\ \underline{\mathbf{p}}_i^{\oplus}(\omega)\end{Bmatrix} = \begin{Bmatrix}\underline{\mathbf{q}}_b(\omega)\\ \underline{\mathbf{q}}_i(\omega)\end{Bmatrix}\ , \tag{27}$$

where the dimension of the vector of incoming pressure waves at the boundary matches the dimension of the vector of the virtual sources $\dim(\underline{\mathbf{p}}_b^{\oplus}(\omega)) = \dim(\underline{\mathbf{q}}_b(\omega)) = n$. Rearrangement of the solution vector, and solving of the resulting equation system, leads to the matrix equation

$$\begin{Bmatrix}\underline{\mathbf{q}}_b(\omega)\\ \underline{\mathbf{p}}_i^{\oplus}(\omega)\end{Bmatrix} = \begin{bmatrix}\mathbf{S}_{bb}(\omega) - \mathbf{S}_{bi}(\omega)\mathbf{S}_{ii}^{-1}(\omega)\mathbf{S}_{ib}(\omega) & \mathbf{S}_{bi}(\omega)\mathbf{S}_{ii}^{-1}(\omega)\\ -\mathbf{S}_{ii}^{-1}(\omega)\mathbf{S}_{ib}(\omega) & \mathbf{S}_{ii}^{-1}(\omega)\end{bmatrix}\begin{Bmatrix}\underline{\mathbf{p}}_b^{\oplus}(\omega)\\ \underline{\mathbf{q}}_i(\omega)\end{Bmatrix}\ . \tag{28}$$

The solution of the incoming pressure waves at the inner nodes $\underline{\mathbf{p}}_i^{\oplus}(\omega)$, as a function of the pressure waves incoming at the boundary nodes $\underline{\mathbf{p}}_b^{\oplus}(\omega)$ and of the inner sources $\underline{\mathbf{q}}_i(\omega)$, can be substituted in the partitioned scattering matrix of the network (20)

$$\begin{Bmatrix}\underline{\mathbf{p}}_b^{\ominus}(\omega)\\ \underline{\mathbf{p}}_i^{\ominus}(\omega)\end{Bmatrix} = \begin{bmatrix}\mathbf{G}_{bb}(\omega) & \mathbf{G}_{bi}(\omega)\\ \mathbf{G}_{ib}(\omega) & \mathbf{G}_{ii}(\omega)\end{bmatrix}\begin{Bmatrix}\underline{\mathbf{p}}_b^{\oplus}(\omega)\\ \underline{\mathbf{p}}_i^{\oplus}(\omega)\end{Bmatrix} + \begin{Bmatrix}\underline{\mathbf{q}}_b^{\ominus}(\omega)\\ \underline{\mathbf{q}}_i^{\ominus}(\omega)\end{Bmatrix}\ , \tag{29}$$

to obtain the scattering matrix of a complex element

$$\mathbf{G}(\omega) = \mathbf{G}_{bb}(\omega) - \mathbf{G}_{bi}(\omega)\mathbf{S}_{ii}^{-1}(\omega)\mathbf{S}_{ib}(\omega) \tag{30}$$

with the source vector

$$\underline{\mathbf{q}}^{\ominus}(\omega) = \mathbf{G}_{bi}(\omega)\mathbf{S}_{ii}^{-1}(\omega)\underline{\mathbf{q}}_i(\omega) + \underline{\mathbf{q}}_b^{\ominus}(\omega)\ . \tag{31}$$

The derivation of the transformation matrix $\mathbf{T}(\omega)$ and the transformed source vector $\underline{\mathbf{q}}(\omega)$ completes the class interface for a complex element. Partitioning the transformation matrix of the complete network (21) leads to

$$\begin{Bmatrix} \underline{\mathbf{s}}_b(\omega) \\ \underline{\mathbf{s}}_i(\omega) \end{Bmatrix} = \begin{bmatrix} \mathbf{T}_{bb}(\omega) & \mathbf{T}_{bi}(\omega) \\ \mathbf{T}_{ib}(\omega) & \mathbf{T}_{ii}(\omega) \end{bmatrix} \begin{Bmatrix} \underline{\mathbf{p}}_b^{\oplus}(\omega) \\ \underline{\mathbf{p}}_i^{\oplus}(\omega) \end{Bmatrix} + \begin{Bmatrix} \underline{\mathbf{q}}_{g,b}(\omega) \\ \underline{\mathbf{q}}_{g,i}(\omega) \end{Bmatrix} . \tag{32}$$

Substituting the incoming pressure waves at the inner nodes $\underline{\mathbf{p}}_i^{\oplus}(\omega)$ yields the transformation matrix of a complex element

$$\mathbf{T}(\omega) = \mathbf{T}_{bb}(\omega) - \mathbf{T}_{bi}(\omega)\mathbf{S}_{ii}^{-1}(\omega)\mathbf{S}_{ib}(\omega) \tag{33}$$

with the associated source vector

$$\underline{\mathbf{q}}(\omega) = \mathbf{T}_{bi}(\omega)\mathbf{S}_{ii}^{-1}(\omega)\underline{\mathbf{q}}_i(\omega) + \underline{\mathbf{q}}_{g,b}(\omega) . \tag{34}$$

CLOSURE

The methodology presented extends the Glav and Åbom [4] formulation to include distributed and lumped multi-port elements. A consistent method has been presented for integrating lumped 2-port elements (such as area discontinuities) and lumped multi-port elements (such as branching) as nodes of the acoustic network; a treatment that minimizes the total number of system unknowns. For cascades of 2-port elements, it is anticipated that the projection matrix $\mathbf{\Pi}$ and the system matrix $\mathbf{S}$ can be adjusted for improved performance. Alternatively, the inversion of the submatrix $\mathbf{S}_{ii}$ could be accelerated.

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