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FREE VIBRATIONS OF CYLINDRICAL SHELLS WITH GENERAL BOUNDARY CONDITIONS

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Abstract
In this paper, a general analysis method is developed for the free vibrations of cylindrical shells with general elastic supports. It is shown that the classical simple boundary conditions can be simply treated as the special cases when the stiffness constants for the restraining springs become either zero or infinity. The shell displacements are expanded as an improved Fourier series, which ensures the excellent accuracy and numerical stability of the solutions. This method is potentially advantageous for the mid-frequency vibration analysis because of its analytical forms of the solutions and involvement of much fewer degrees of freedom in comparison with the traditional grid-based methods.

INTRODUCTION

Shells together with beams and plates are the structural elements most commonly used in industrial applications. Vibrations of shells have been an active research subject in structural dynamics for many years, resulting in a large number of publications. For instance, there are approximately 1000 research publications that have been referred in Leissa's book [1]. In comparison with beams and plates, the shell problems are far more complicated. Many shell theories have been used to include the various effects associated with shell deformations or stress components. Unlike beams and plates, the coupling between different displacement components is usually important due to the curvatures of the shells. Accordingly, the dynamic characteristic of a shell tends to become more sensitive to the boundary conditions.

In the literature, there are many investigations about the free vibrations of shells under various boundary conditions. Each kind of boundary conditions will typically require a specific analysis method or solution procedure. Unfortunately, even consider the so-called simple boundary conditions at each end, they constitute to 136 different combinations for a shell. This difficulty will become much more remarkable if a shell is elastically restrained at ends. To avoid unnecessary complications, the simply supported boundary condition is often considered as default in many applications such as sound radiation from shells. However, the boundary conditions encountered in real world can be significantly different from the simply-supported.

In this paper, a general method is presented for the vibration analysis of shell with elastic boundary supports. More importantly, this method can be universally applied to shells with various boundary conditions. Varying the boundary condition will simply require a change of the stiffness constants for the restraining springs just like modifying other geometrical or material parameters such as shell thickness or mass density. The excellent accuracy and convergence of the solution are demonstrated through numerical examples.

FORMULATIONS

Consider an elastically restrained circular cylindrical shell of radius R, thickness h and length L. Let u, v, and w denote the displacements in the axial x, circumferential θ and radial r directions, respectively. The equations of the motions of the shell can then be written as

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{x\theta}}{R\partial\theta} = \rho h \frac{\partial^2 u}{\partial t^2} \tag{1}$$

$$\frac{\partial N_{x\theta}}{\partial x} + \frac{\partial N_{\theta}}{R\partial\theta} = \rho h \frac{\partial^2 v}{\partial t^2} \tag{2}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_{\theta}}{R\partial\theta} - \frac{N_\theta}{R} = \rho h \frac{\partial^2 w}{\partial t^2} \tag{3}$$

where N_x, $N_{x\theta}$, N_θ, Q_x and Q_θ represent the resultant forces acting on the mid-surface. The boundary conditions can be accordingly expressed as:

at x=0,

$$N_x - k_1 u = 0 \quad , \quad N_{x\theta} - k_2 v = 0 \tag{4, 5}$$

$$Q_x + \frac{\partial M_{x\theta}}{R\partial\theta} - k_3 w = 0, \qquad M_x + k_4 \frac{\partial w}{\partial x} = 0; \tag{6, 7}$$

at x=L,

$$N_x + k_5 u = 0 \quad , \quad N_{x\theta} + k_6 v = 0 \tag{8, 9}$$

$$Q_x + \frac{\partial M_{x\theta}}{R\partial\theta} + k_7 w = 0 \quad , \qquad M_x - k_8 \frac{\partial w}{\partial x} = 0 \tag{10, 11}$$

where k_1, k_2,.., k_8 are the stiffness constants for the restraining springs.

The resultant forces and moments in the above equations are the functions of the displacement components, and the expressions are readily found in any book about shells. It should be note that the elastic supports at ends represent a set of general boundary conditions; all the simple boundary conditions can be considered as its special case when the spring constants take some extreme values such as zero or infinity.

In this study, the displacements will be sought in the following forms

$$u(x,\theta) = \left(\sum_{m=0}^{\infty} a_m \cos \lambda_m x + p_u(x) \right) \cos n\theta, \qquad (\lambda_m = \frac{m\pi}{L}) \tag{12}$$

$$v(x,\theta) = \left(\sum_{m=0}^{\infty} b_m \cos \lambda_m x + p_v(x) \right) \sin n\theta, \tag{13}$$

$$w(x,\theta) = \left(\sum_{m=0}^{\infty} c_m \cos \lambda_m x + p_w(x) \right) \cos n\theta \tag{14}$$

In Eqs. (12-14), $p_u(x)$, $p_v(x)$ and $p_w(x)$ represent a set of closed form functions, satisfying:

$$\left.\frac{\partial p_u(x)}{\partial x}\right|_{x=0} = \left.\frac{\partial u(x,0)}{\partial x}\right|_{x=0} = \beta_1 , \qquad \left.\frac{\partial p_u(x)}{\partial x}\right|_{x=L} = \left.\frac{\partial u(x,0)}{\partial x}\right|_{x=L} = \beta_2 \tag{15, 16}$$

$$\left.\frac{\partial p_v(x)}{\partial x}\right|_{x=0} = \left.\frac{\partial v(x,\pi/2)}{\partial x}\right|_{x=0} = \beta_3 , \qquad \left.\frac{\partial p_v(x)}{\partial x}\right|_{x=L} = \left.\frac{\partial v(x,\pi/2)}{\partial x}\right|_{x=L} = \beta_4 , \tag{17, 18}$$

$$\left.\frac{\partial p_w(x)}{\partial x}\right|_{x=0} = \left.\frac{\partial w(x,0)}{\partial x}\right|_{x=0} = \beta_5 , \qquad \left.\frac{\partial p_w(x)}{\partial x}\right|_{x=L} = \left.\frac{\partial w(x,0)}{\partial x}\right|_{x=L} = \beta_6 , \tag{19, 20}$$

$$\left.\frac{\partial^3 p_w(x)}{\partial x^3}\right|_{x=0} = \left.\frac{\partial^3 w(x,0)}{\partial x^3}\right|_{x=0} = \beta_7 , \qquad \left.\frac{\partial^3 p_w(x)}{\partial x^3}\right|_{x=L} = \left.\frac{\partial^3 w(x,0)}{\partial x^3}\right|_{x=L} = \beta_8 . \tag{21, 22}$$

It is easy to verify that the above conditions are readily ensured by choosing

$$p_u(x) = \zeta_1(x)\beta_1 + \zeta_2(x)\beta_2 , \qquad p_v(x) = \zeta_1(x)\beta_3 + \zeta_2(x)\beta_4 , \qquad \text{and} \tag{23, 24}$$

$$p_w(x) = \zeta_1(x)\beta_5 + \zeta_2(x)\beta_6 + \zeta_3(x)\beta_7 + \zeta_4(x)\beta_8 . \tag{25}$$

where

$$\begin{Bmatrix} \zeta_1(x) \\ \zeta_2(x) \\ \zeta_3(x) \\ \zeta_4(x) \end{Bmatrix} = \begin{Bmatrix} (6Lx - 2L^2 - 3x^2)/6L \\ (3x^2 - L^2)/6L \\ -(15x^4 - 60Lx^3 + 60L^2x^2 - 8L^4)/360L \\ (15x^4 - 30L^2x^2 + 7L^4)/360L \end{Bmatrix} . \tag{26}$$

Although the auxiliary functions are here chosen as polynomials, they can be actually defined as any closed form functions which are sufficiently smooth over [0, L]. The reason for including these auxiliary functions in the displacement expansions is simply to remove all the possible discontinuities with the displacements and their derivatives at the end points for any given boundary condition. The benefits for doing this have been discussed for the beam and plate problems [2, 3].

By substituting Eqs. (12-25) into the boundary conditions, Eqs. (4-11), the unknown coefficients in the polynomials can be found in terms of Fourier coefficients as

$$\begin{Bmatrix}\beta_1\\ \beta_2\end{Bmatrix} = \sum_{m=0}^{\infty}\mathbf{H}_u^{-1}\mathbf{Q}_u^m a_m , \qquad \begin{Bmatrix}\beta_3\\ \beta_4\end{Bmatrix} = \sum_{m=0}^{\infty}\mathbf{H}_v^{-1}\mathbf{Q}_v^m b_m \quad , \tag{27, 28}$$

$$\{\beta_5 \quad \beta_6 \quad \beta_7 \quad \beta_8\}^T = \sum_{m=0}^{\infty}\mathbf{H}_w^{-1}\mathbf{Q}_w^m c_m . \tag{29}$$

where

$$\mathbf{H}_u = \begin{bmatrix} \frac{\hat{k}_1 L}{3}+1 & \frac{\hat{k}_1 L}{6} \\ \frac{\hat{k}_5 L}{6} & \frac{\hat{k}_5 L}{3}+1 \end{bmatrix}, \quad \mathbf{H}_v = \begin{bmatrix} \frac{\hat{k}_2 L}{3}+\frac{1-\mu}{2} & \frac{\hat{k}_2 L}{6} \\ \frac{\hat{k}_6 L}{6} & \frac{\hat{k}_6 L}{3}+\frac{1-\mu}{2} \end{bmatrix} , \tag{30,31}$$

$$\mathbf{H}_w = \begin{bmatrix} -\frac{\hat{k}_3 L}{3} & -\frac{\hat{k}_3 L}{6} & \frac{\hat{k}_3 L^3}{45}+\kappa & \frac{7\hat{k}_3 L^3}{360} \\ -\frac{\hat{k}_7 L}{6} & -\frac{\hat{k}_7 L}{3} & \frac{7\hat{k}_7 L^3}{360} & \frac{\hat{k}_7 L^3}{45}+\kappa \\ \hat{k}_4+\frac{\kappa}{L} & -\frac{\kappa}{L} & \frac{\kappa L}{3} & \frac{\kappa L}{6} \\ -\frac{\kappa}{L} & \hat{k}_8+\frac{\kappa}{L} & \frac{\kappa L}{6} & \frac{\kappa L}{3} \end{bmatrix} , \tag{32}$$

$$\mathbf{Q}_u^m = \{\hat{k}_1 \quad (-1)^{m+1}\hat{k}_5\}^{\mathrm{T}} , \qquad \mathbf{Q}_v^m = \{\hat{k}_2 \quad (-1)^{m+1}\hat{k}_6\}^{\mathrm{T}} , \tag{33,34}$$

and

$$\mathbf{Q}_w^m = \{-\hat{k}_3 \quad (-1)^m \hat{k}_7 \quad -\kappa\lambda_m^2 \quad (-1)^m \kappa\lambda_m^2\}^T \tag{35}$$

with $\hat{k}_i = k_i / K$ and $K = Eh/(1-\mu^2)$.

In light of Eqs. (27-29), Eqs. (12-14) can be rewritten as

$$u(x,\theta) = \sum_{m=0}^{\infty} a_m \varphi_u^m(x)\cos n\theta \quad , \tag{36}$$

$$v(x,\theta) = \sum_{m=0}^{\infty} b_m \varphi_v^m(x)\sin n\theta \quad , \tag{37}$$

and $$w(x,\theta) = \sum_{m=0}^{\infty} c_m \varphi_w^m(x)\cos n\theta \quad , \tag{38}$$

where

$$\varphi_\alpha^m(x) = \cos\lambda_m x + \zeta(x)^T\,\overline{\mathbf{Q}}_\alpha^m \qquad (\alpha = u, v, w) \tag{39}$$

with

$$\zeta(x)^T = \begin{Bmatrix}\zeta_1(x)\\ \zeta_2(x)\\ \zeta_3(x)\\ \zeta_4(x)\end{Bmatrix}, \qquad \overline{\mathbf{Q}}_\alpha^m = \begin{Bmatrix}\tilde{\mathbf{Q}}_\alpha^m\\ 0\\ 0\end{Bmatrix} \qquad (\alpha = u, v) \tag{40,41}$$

$$\overline{\mathbf{Q}}_w^m = \tilde{\mathbf{Q}}_w^m \quad \text{and} \quad \tilde{\mathbf{Q}}_\alpha^m = (\mathbf{H}_\alpha)^{-1}\mathbf{Q}_\alpha^m \quad . \tag{42,43}$$

The Rayleigh-Ritz method will be employed here to determine the Fourier coefficients. The potential energy consistent with the Donnell-Mushtari theory can be expressed as

$$V = \frac{K}{2}\int_0^L\int_0^{2\pi}\left\{\left(\frac{\partial u}{\partial x}+\frac{\partial v}{R\partial\theta}+\frac{w}{R}\right)^2 - 2(1-\nu)\frac{\partial u}{\partial x}\left(\frac{\partial v}{R\partial\theta}+\frac{w}{R}\right)+\frac{(1-\nu)}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{R\partial\theta}\right)^2 + \right.$$

$$\left.\kappa\left[\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{R^2\partial\theta^2}\right)^2 - 2(1-\nu)\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{R^2\partial\theta^2}-\left(\frac{\partial^2 w}{R\partial x\partial\theta}\right)^2\right)\right]\right\}Rdxd\theta$$

$$+1/2\int_0^{2\pi}[k_1 u^2 + k_2 v^2 + k_3 w^2 + k_4(\partial w/\partial x)^2]_{x=0}Rd\theta$$

$$+1/2\int_0^{2\pi}[k_5 u^2 + k_6 v^2 + k_7 w^2 + k_8(\partial w/\partial x)^2]_{x=L}Rd\theta \quad . \tag{44}$$

The total kinetic energy is calculated from

$$T = \frac{1}{2}\int_0^L\int_0^{2\pi}\rho h\left[(\partial u/\partial t)^2 + (\partial v/\partial t)^2 + (\partial w/\partial t)^2\right]Rdxd\,\theta\,. \tag{45}$$

By making use of Eqs. (44) and (45), the Rayleigh-Ritz procedure will lead to a final system equation for the Fourier coefficients:

$$\begin{bmatrix}\Lambda^{ss} & \Lambda^{s\theta} & \Lambda^{sr}\\ \Lambda^{s\theta\,\mathrm{T}} & \Lambda^{\theta\theta} & \Lambda^{\theta r}\\ \Lambda^{sr\,\mathrm{T}} & \Lambda^{\theta r\,\mathrm{T}} & \Lambda^{rr}\end{bmatrix}\begin{Bmatrix}\overline{\mathbf{a}}\\ \overline{\mathbf{b}}\\ \overline{\mathbf{c}}\end{Bmatrix} - \omega^2\begin{bmatrix}\mathbf{M}^{ss} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}^{\theta\theta} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{M}^{rr}\end{bmatrix}\begin{Bmatrix}\overline{\mathbf{a}}\\ \overline{\mathbf{b}}\\ \overline{\mathbf{c}}\end{Bmatrix} = 0 \tag{46}$$

where

$$\overline{\mathbf{a}} = \{a_{10}, a_{11}, \ldots, a_{mn}, \ldots, a_{MN}\}^{\mathrm{T}}, \tag{47}$$

$$\overline{\mathbf{b}} = \{b_{11}, b_{12}, \ldots, b_{mn}, \ldots, b_{MN}\}^{\mathrm{T}}, \tag{48}$$

$$\overline{\mathbf{c}} = \{c_{10}, c_{11}, \ldots, c_{mn}, \ldots, c_{MN}\}^{\mathrm{T}}, \tag{49}$$

$$\Lambda^{ss}_{mn,m'n'} = \delta_{nn'}[I^{mm'}_{uu,11} + \frac{(1-\mu)n^2}{2R^2}I^{mm'}_{uu,00} + \frac{2}{L}\hat{k}_1\varphi_u^m(0)\varphi_u^{m'}(0) + \frac{2}{L}\hat{k}_5\varphi_u^m(L)\varphi_u^{m'}(L)], \tag{50}$$

$$\Lambda^{s\theta}_{mn,m'n'} = \delta_{nn'}[\frac{\mu n}{R}I^{mm'}_{uv,10} - \frac{(1-\mu)n}{2R}I^{mm'}_{uv,01}], \tag{51}$$

$$\Lambda^{sr}_{mn,m'n'} = \delta_{nn'}\frac{\mu}{R}I^{mm'}_{uw,10}\,, \tag{52}$$

$$\Lambda^{\theta\theta}_{mn,m'n'} = \delta_{nn'}[\frac{n^2}{R^2}I^{mm'}_{vv,00} + \frac{(1-\mu)}{2}I^{mm'}_{vv,11} + \frac{2}{L}\hat{k}_2\varphi_v^m(0)\varphi_v^{m'}(0) + \frac{2}{L}\hat{k}_6\varphi_v^m(L)\varphi_v^{m'}(L)], \tag{53}$$

$$\Lambda^{\theta r}_{mn,m'n'} = \delta_{nn'}\frac{n}{R^2}I^{mm'}_{vw,00}\,, \tag{54}$$

$$\Lambda^{\theta\theta}_{mn,m'n'} = \delta_{nn'}\{\frac{1}{R^2} I^{mm'}_{ww,00} + \kappa[I^{mm'}_{ww,22} + \frac{n^4}{R^4} I^{mm'}_{ww,00} + 2(1-\mu)\frac{n^2}{R^2} I^{mm'}_{ww,11}$$
$$- \frac{\mu n^2}{R^2}(I^{mm'}_{ww,02} + I^{mm'}_{ww,20})] + \frac{2}{L}\hat{k}_3 \varphi^m_w(0)\varphi^{m'}_w(0) + \frac{2}{L}\hat{k}_7 \varphi^m_w(L)\varphi^{m'}_w(L) + \tag{55}$$
$$+ \frac{2}{L}\hat{k}_4 \frac{\partial \varphi^m_w(0)}{\partial x}\frac{\partial \varphi^{m'}_w(0)}{\partial x} + \frac{2}{L}\hat{k}_8 \frac{\partial \varphi^m_w(L)}{\partial x}\frac{\partial \varphi^{m'}_w(L)}{\partial x}\}$$

$$\mathbf{M}^{ss}_{mn,m'n'} = \delta_{nn'}\hat{\rho} h I^{mm'}_{uu,00}\ , \qquad \mathbf{M}^{\theta\theta}_{mn,m'n'} = \delta_{nn'}\hat{\rho} h I^{mm'}_{vv,00}\ , \tag{56}$$

$$\mathbf{M}^{rr}_{mn,m'n'} = \delta_{nn'}\hat{\rho} h I^{mm'}_{ww,00} \qquad (\hat{\rho} = \rho / K)\ , \tag{57}$$

and $$I^{mm'}_{\alpha\beta,pq} = 2/L\int_0^L \frac{\partial^p \varphi^m_\alpha}{\partial x^p}\frac{\partial^q \varphi^{m'}_\beta}{\partial x^q}dx \qquad (\alpha,\beta = u,v,w)\ . \tag{58}$$

Equation (46) represents a characteristic equation for a matrix eigenproblem from which the eigenvalues and eigenvectors are readily determined. It should be noted that the components in each eigenvector are actually the Fourier expansion coefficients for the corresponding mode whose physical shape can be directly calculated from Eqs. (36-38). Obviously, in actual numerical calculations, the displacement expansions will need to be truncated to include only a finite number of terms.

RESULTS AND DISCUSSIONS

As an example, we consider a cylindrical shell that is simply-supported at each end. The simply supported boundary condition is specified as: $N_x = M_x = v = w = 0$. In term of the restraining springs, this boundary condition can be easily achieved by letting $k_{2,6} = k_{3,7} = \infty$ and $k_{1,5} = k_{4,8} = 0$ (the infinite stiffness is actually represented by a large number, 10^{10}, in actual calculations).

Table 1. Frequency parameters, $\Omega = \omega R\sqrt{\rho(1-\mu^2)/E}$, for a simply-supported shell: a=4R , h/R=0.05 and μ=0.3.

Mode		$\Omega = \omega R\sqrt{\rho(1-\mu^2)/E}$				
		0	1	2	3	4
m=1,	*Current*	0.464648	0.257385	0.127128	0.143327	0.234822
	Exact	0.464648	0.257385	0.127128	0.143327	0.234822
m=2,	*Current*	0.928907	0.574179	0.337652	0.248813	0.285620
	Exact	0.928907	0.574176	0.337649	0.248810	0.285619
m=3,	Current	0.948172	0.764375	0.532951	0.399893	0.383688
	Exact	0.948172	0.764355	0.532923	0.399865	0.383667

The results in Table 1 are calculated by truncating the series expansions at M=10. To example the accuracy and convergence of the solution, Table 2 compares the frequency parameters, $\Omega = \omega R\sqrt{\rho(1-\mu^2)/E}$, obtained using various numbers of terms in the series expansions. The excellent accuracy and convergence of the current solution are evident from these results.

Table 2. The frequency parameters, $\Omega = \omega R\sqrt{\rho(1-\mu^2)/E}$, obtained using various numbers of terms in the displacement expansions.

Number of terms used in the series	$\Omega = \omega R\sqrt{\rho(1-\mu^2)/E}$				
	0	1	2	3	4
M=5	0.464652	0.257389	0.127132	0.143329	0.234823
M=7	0.464649	0.257386	0.127129	0.143327	0.234822
M=9	0.464648	0.257385	0.127128	0.143327	0.234822
M=10	0.464648	0.257385	0.127128	0.143327	0.234822

Next, consider a cylindrical shell clamped at both ends. The clamped condition means: $u = v = w = \partial w/\partial x = 0$. Under the current method, the clamped-clamped boundary conditions are considered the same as the simply-supported except that the stiffness constants for the restraining springs now all become infinitely large. The dimensions for the shell are as follows: a=0.502 m, R=0.0635 m and h=0.00163 m. The material properties are chosen as: E=2.1×10^{11}, μ=0.28 and ρ=7800. Listed in Table 3 are the natural frequencies for some lower-order modes.

Table 3. The natural frequencies for a clamped-clamped shell.

Mode	*Current m=1*	*Ref. [4]* *	*Current m=2*	*Ref. [4]*
n=1	1886.74	2035.05	3854.75	4302.05
2	934.220	971.531	2039.66	2189.59
3	982.265	990.339	1454.80	1500.07
4	1598.55	1600.90	1769.54	1782.28
5	2484.78	2486.49	2572.31	2578.07

* Note: Calculated using Eq. (12-26) on page 310 of Ref. [4].

Now let us assume that in the above example the clamped condition at x=a is modified to being free while the left end remains the same, that is, $k_1 = k_2 = k_3 = k_4 = \infty$ and $k_5 = k_6 = k_7 = k_8 = 0$. The natural frequencies for some of lower order modes are given in Table 4.

It should be noted that although meaningful differences are observed for certain frequency pairs, the comparisons are generally considered satisfactory in view that the solution schemes are considerably different. In the last two examples, the series expansions have been truncated to M=15.

Table 4. Natural frequencies for a clamped-Free shell.

Mode	*Current m=1*	*Ref. [4]*	*Current m=2*	*Ref. [4]*
n=1	508.860	505.513	2109.47	2497.59
2	413.377	399.676	1049.11	1051.41
3	867.385	865.162	1038.03	1016.06
4	1563.38	1563.47	1615.02	1622.82

CONCLUSIONS

A general method is developed for vibration analyses of cylindrical shells with general boundary restraints. This method offers a unified solution for various boundary conditions. Different boundary conditions can be effectively created by simply modifying the stiffness constants for the restraining springs. Since the displacements on a shell is represented in terms of a complete set of closed-formed basis functions, this method will involve much smaller number of degrees of freedom as compared with the grid-based methods such as FEM. In addition, the sinusoidal basis functions will better capture the wave behavior of the structural vibration at higher frequencies.

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