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ILU Preconditioners for Solving Three-Dimensional Helmholtz Equation

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Abstract

Modelling sound propagation over large domains presents severe challenges with respect to computational resources. In general, the direct solutions of the system equations arising from the full field discretization for problems of any significant size of practical engineering interest cannot be attempted. The present study proposes an efficient iterative solution of a three-dimensional Helmholtz equation. The discretization is done using a Wave Based Finite Difference scheme known as the Wave Expansion Method (WEM). The WEM requires only around 2-3 nodes per wavelength to obtain accurate solutions which offers a significant improvement upon conventional Finite Element/Finite Difference techniques, which require approximately 8-10 nodes per wavelength.

INTRODUCTION

Modelling sound propagation for problems of practical engineering interest have always been a difficult task due to the amount of computational costs associated with them. Several methods for the solution of the three dimensional Helmholtz equation have been proposed by various researchers [6, 7, 8].

The conventional acoustics analysis methods such as the Finite Differences (FDM), the Finite Element (FEM) and the Boundary Element techniques consider the theoretical nature of wave propagation but at a relatively high computational cost. These techniques require the whole domain to be discretized with a smooth mesh in order to obtain accurate solutions.

In this study, a suitable iterative solver is explored to run in conjunction with the Wave Expansion Method which offers a distinct advantage over conventional FE/FD techniques in terms of discretizational efficiency.

Acoustic Problem Definition

The propagation of harmonic acoustic waves is governed by the Helmholtz equation [9]

$$\nabla^2 p + k^2 p = 0 \tag{1}$$

The ratio

$$k = \frac{w}{c} \tag{2}$$

is the wavenumber and w=2πf is the circular frequency.

The performance of the numerical system is assessed by the number of nodes per wavelength required by the discretization to produce accurate results. The number of nodes per wavelength is given by Eq. 3

$$ppw = \frac{c}{h \cdot f} \tag{3}$$

where h is the maximum nodal spacing in the mesh.

Wave Expansion Method

As mentioned previously, the cost of numerical methods is associated with the number of nodes per wavelength required for the accurate discretization of the problem [11]. The Wave Expansion Method solves the Helmholtz Equation by domain discretization i.e. relating the value of an unknown discrete point of the domain to a set of neighboring points. Basically, the method provides an interpolation procedure for Eq. 1 using plane waves as fundamental solutions. This method is primarily derived from the Green's Function Discretization method developed by Caruthers *et al*. [3]. The method is very flexible and can be formulated on both structured and unstructured meshes [2]. For a detailed description about the Wave Expansion Method and the boundary conditions implementation the reader is referred to [4, 5, 11, 12].

Conventional FE/FD methods require approximately around 8-10 nodes per wavelength to achieve accurate solutions [17]. The value of the WEM [3, 4, 5, 12] lies in the fact that it requires only between 2-3 nodes per wavelength to achieve accurate and convergent solutions.

Solution Procedure

Any numerical technique is incomplete without an adequate solution procedure. Any type of discretization assembles the equations in the form as shown in Eq. 4

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{4}$$

where A is the coefficient matrix, x is the solution vector and b is determined by the boundary data.

The coefficient matrix A, obtained by the WEM is sparse, unsymmetric and complex. The solution of the equations by a direct elimination method utilizes a lot of memory and CPU requirements, especially in three-dimensions. This is due to a large 'fill-in' that can overwhelm the capacity of the largest computer resources that currently exist. Moreover, the indefiniteness of the matrix Eq. 4 at large wavenumbers presents a major difficulty when attempting the procedure with iterative solvers.

Hence, we resort to finding a solution by employing a Krylov subspace method [1, 13, 16, 15] with a suitable preconditioner. In general, in a preconditioned system, Eq. 5 is solved rather than Eq. 4.

$$\mathbf{AM^{-1}y = b} \tag{5}$$

where

$$\mathbf{x = M^{-1}y} \tag{6}$$

In this study, we have employed the BI-CGSTAB method as described by Van der vorst [16] as an iterative solver. The preconditioner here is constructed by a 'dual threshold strategy' as described by Saad [14].

However, a significant disadvantage with regard to using the 'dual threshold strategy' algorithm is that it is not an easy task to predict the accurate 'drop tolerance' value [10, 14]. These values are highly problem dependent and are usually acquired through a 'trial and error' approach with a small number of sample matrices. In our study, these values are obtained after tests based on the sample matrices obtained after discretization while bearing in mind the memory requirements posed due to the 'fill-in' obtained during the LU factorization.

Results are presented for a set of problems discretized using the WEM. In the simulations, the the zero vector x_0 is the initial guess and iterations are carried out until the residual vector attains a tolerance level of 10^{-3}. The equation solver has been implemented in Matlab using double precision accuracy. The problems reported herein were solved on an IBM, Pentium-IV machine with 882 Mb of RAM.

The sparsity pattern of the matrix obtained by the discretization of the Helmholtz Equation by the Wave Expansion Method on structured meshes is shown in Fig. 1 while Table 1 presents the performance of the numerical system using the WEM.

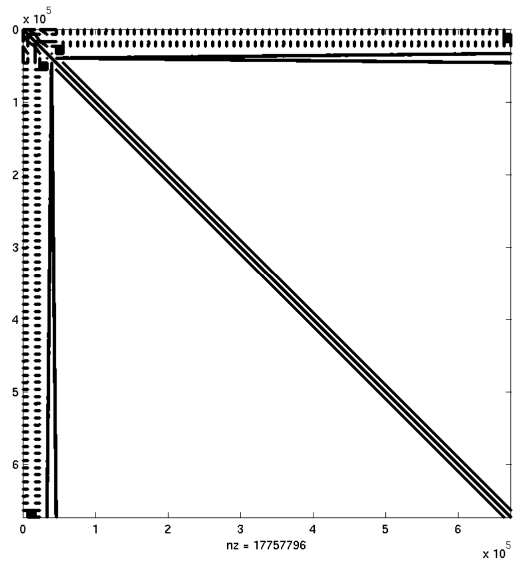

Figure 1: Sparsity Pattern of the stiffness Matrix discretized with a structured mesh

Table 1: Performance of the Numerical System with the preconditioner

S.No	No.of nodes	λ/h	freq (Hz)	iter	CPU time (s)	Cost (MB)
1	63756	3	100	332	233.72	56
2	78608	3	4000	437	369	65
3	171666	3	12000	626	1201.4	108
4	375821	3	12000	843	3630.43	271
5	1030301	3	12000	1184	13395.54	583

The same set of problems have been studied without the usage of the preconditioner and the results have been presented in Table 2.

Table 2: Performance of the Numerical System without the preconditioner

S.No	No.of nodes	λ/h	freq (Hz)	iter	CPU time (s)	Cost (MB)
1	63756	3	100	333.5	245.81	49
2	78608	3	4000	437	367.73	54
3	171666	3	12000	625.5	1198.04	91
4	375821	3	12000	845	3617.21	237
5	1030301	3	12000	1195	14330.34	545

Fig. 2 shows the sound pressure distribution obtained on a cube of 1m. The total number of equations was 1030301 and the domain was discretized using a frequency of 12000 Hz and a mesh spacing of 0.01m, thus giving an overall nodal density of around 3 nodes per wavelength.

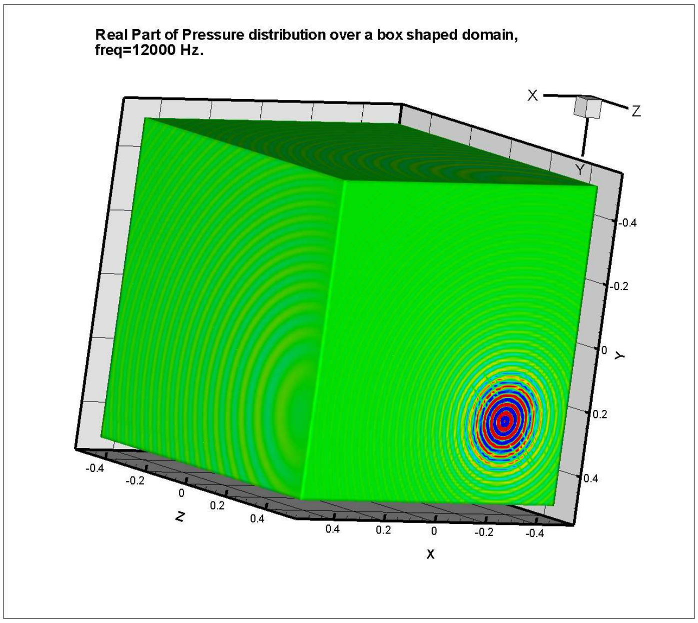

Figure 2: Sound Pressure distribution for a box shaped domain

Summary

This paper considers the usage of the Wave Expansion Method for modelling sound propagation of problems of real engineering interest. The method proves to be efficient for a wide range of frequencies (100 Hz to 12,000 Hz). It also proves to be stable for various types of boundary conditions and high frequency scattering problems [2]. The ultimate effectiveness of the WEM not only depends on the discretization procedure used but also on the efficiency of the solution techniques. The results presented here show that the WEM is highly efficient computational technique particularly when compared with conventional FE/FD and BEM techniques. The results shown in Table 2 indicate that the method is suitably efficient enough to not press the demand for the construction of a preconditioner which can be an expensive task. By using the BI-CGSTAB algorithm the scope for parallelism is also realized.

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