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NUMERICAL SIMULATION OF STREAMING IN RESONATOR BASED THERMOACOUSTIC DEVICES

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Abstract

This paper is focused on efficient numerical simulation of secondary streaming arising inside an acoustic-resonance tube refrigerator. The model that is derived in this work is based on the boundary-layer approximation of Navier-Stokes equations. Viscous losses near the walls are taken into account within the framework of the FDTD (finite-difference time-domain) discretization of the acoustic equations using a subgrid-scale analytical approximation for the boundary layers. The obtained quantities of the first (acoustic) approximation enter as nonlinear source terms the equations for the second order approximation and thus trigger second harmonics and static fields. Numerical simulation of the second order equation results in acoustic streaming in the standing-wave thermoacoustic engine.

INTRODUCTION

Although one has already a fairly clear idea of the functioning of thermo-acoustic devices [4], the number of sufficiently precise and general analytical and numerical models is limited. Such models are nevertheless required to allow for optimization and extension of the field of application. A particular problem is caused by nonlinear acoustic phenomena which play an important part in thermo-acoustic devices, operated near a resonance. Nonlinear effects include effects of second order such as “acoustic streaming”.

“Acoustic streaming” leads to a mass transport which does not vanish in the time. For this effect, which can both reinforce or reduce the basic thermo-acoustic effect, analytical models were developed for specific situations and with particular approaches (to mention only one of them [1]). The applicability of analytical models is limited to resonators of thermo-acoustic devices or to the space between the plates of stack. Such analytical models can however be used as a sub-model in a numerical model.

An approach that used analytical results for the acoustic field in numerical simulation of streaming through the whole device [3] required accurate interface condition in the cross-section separating the stack and the tube-resonator.

In the present work, acoustic streaming in a thermo-acoustic tube-refrigerator is simulated in two optimized steps: a numerical simulation of the acoustic field followed by a numerical simulation of second order streaming. For this purpose, the FDTD (finite-difference time-domain) discretization of the acoustic equations had to be extended with a subgrid-scale analytical approximation for the boundary layers. The acoustic fields are used for deriving source terms that are responsible for acoustic streaming. Detailed numerical simulation of streaming is performed using optimal code for incompressible fluid dynamics.

MODIFIED NAVIER-STOKES EQUATIONS FOR A BOUNDARY LAYER

To solve the problem in acoustic approximation the Finite-Difference-Time-Domain (FDTD) method [2] can be used. It discretizes the set of governing equation in time domain. The very different length scales of visco-thermal effects near boundaries and acoustic wavelength however impose strong requirements on the numerical approach. Since it can safely be assumed that the boundary effects are localized within a layer that is very thin compared to the acoustic wavelength, a subgrid-scale approximation is introduced to account for visco-thermal effects during the acoustic simulation. This approximation implies that for cells contained a boundary layer, Navier-Stokes equations are solved analytically and the effect on the acoustic wave is averaged over the cross-section of a cell. For the rest of fluid, where the viscous effects are negligible in comparison with the latter within a boundary layer, linearized Euler equations are solved. Equations describing acoustic oscillation in a viscous and heat conducting medium in the boundary layer approximation have the form (in 2D):

$$\begin{aligned}
&\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2},\\
&\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(v\rho)}{\partial y}=0,\\
&\rho c_p\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=\lambda\frac{\partial^2 T}{\partial y^2}+\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x},\\
&p=\rho R^*T,\ R^*=c_p-c_v
\end{aligned}\tag{1}$$

where x and y are the longitudinal and transverse coordinates; u and v are the longitudinal and transverse components of velocity, respectively; T is the temperature; R^* is the universal gas constant; c_p and c_v are the specific heats at constant pressure and volume, respectively; μ is the dynamic viscosity coefficient; λ

is the heat conductivity coefficient; t is time.
Let us multiply the second equation by u and add it to the first. One obtains:

$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u^2)}{\partial x}+\frac{\partial(\rho uv)}{\partial y}=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2} \tag{2}$$

After multiplying (2) by $1/dy$ (where $\delta=\sqrt{2\nu/\omega}$ is the boundary layer thickness, ν is the kinematic viscosity coefficient, ω is the cyclic frequency of oscillation) and integrating over y from $h-dy$ to h (or from $-h$ to $-h+dy$), where the boundary is at y=h (or y=-h), it becomes

$$\frac{\partial(\overline{\rho u})}{\partial t}+\frac{\partial(\overline{\rho u^2})}{\partial x}+\frac{\partial(\overline{\rho uv})}{\partial y}=-\frac{\partial p}{\partial x}-\frac{1}{dy}\tau_W \tag{3}$$

where

$$\frac{\partial(\overline{\rho u})}{\partial t}=\frac{1}{dy}\int_{h-dy}^{h}\frac{\partial(\rho u)}{\partial t}dy,\quad \frac{\partial(\overline{\rho u^2})}{\partial t}=\frac{1}{dy}\int_{h-dy}^{h}\frac{\partial(\rho u^2)}{\partial t}dy,$$

$$-\frac{1}{dy}\tau_W=\frac{\mu}{dy}\int_{h-dy}^{h}\frac{\partial^2 u}{\partial y^2}dy,\quad \frac{\partial(\overline{\rho uv})}{\partial y}=\frac{1}{dy}\int_{h-dy}^{h}\frac{\partial(\rho u\mathbf{v})}{\partial y}dy=-\frac{1}{dy}\rho uv\big|_{h-dy}.$$

It will be shown below that τ_W is approximately the tangential stress at the wall.
Similarly, multiplying the second equation of (1) by c_pT and adding it to the third yields

$$c_p\frac{\partial(\rho T)}{\partial t}+c_p\frac{\partial(\rho uT)}{\partial x}+c_p\frac{\partial(v\rho T)}{\partial y}=\lambda\frac{\partial^2 T}{\partial y^2}+\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x} \tag{4}$$

Making use of equation of state and substituting $\rho T=p/R^*$ in (4) yields (if $c_p=const$)

$$\frac{c_p}{R^*}\left(\frac{\partial p}{\partial t}+\frac{\partial(up)}{\partial x}+\frac{\partial}{\partial y}(vp)\right)=\lambda\frac{\partial^2 T}{\partial y^2}+\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x} \tag{5}$$

After the procedure of averaging over the cross-section of a cell, one obtains

$$\frac{\partial\overline{p}}{\partial t}+\gamma\overline{p}\frac{\partial\overline{u}}{\partial x}+\overline{u}\frac{\partial\overline{p}}{\partial x}+\frac{\partial(\overline{vp})}{\partial y}=\frac{q_w}{dy} \tag{6}$$

where

$$\frac{\partial\overline{p}}{\partial t}=\frac{1}{dy}\int_{h-dy}^{h}\frac{\partial p}{\partial t}dy,\quad \overline{p}\frac{\partial\overline{u}}{\partial x}+\overline{u}\frac{\partial p}{\partial x}=\frac{1}{dy}\int_{h-dy}^{h}\frac{\partial(up)}{\partial x}dy,$$

$$\frac{q_W}{dy}=\frac{\lambda}{dy}\int_{h-dy}^{h}\frac{\partial^2 T}{\partial y^2}dy,\quad \frac{\partial(\overline{vp})}{\partial y}=\frac{1}{dy}\int_{h-dy}^{h}\frac{\partial}{\partial y}(vp)dy=-\frac{1}{dy}vp\big|_{h-dy}$$

where q_W is the heat flux at the wall, $\kappa=c_p/c_v$.

Thus, the set of equations describing gas oscillation in a viscous heat conducting layer averaged over the cell is

$$\frac{\partial\left(\overline{\rho u}\right)}{\partial t}+\frac{\partial\left(\overline{\rho u^2}\right)}{\partial x}+\rho uv\Big|_{h-dy}=-\frac{\partial\overline{p}}{\partial x}-\frac{1}{dy}\tau_W$$
$$\frac{\partial\overline{p}}{\partial t}+\kappa p\frac{\partial\overline{u}}{\partial x}+\overline{u}\,\frac{\partial\overline{p}}{\partial x}+vp\Big|_{h-dy}=\frac{1}{dy}q_W \tag{7}$$

Let us use the method of successive approximations to obtain equations containing terms of acoustic approximation (first order) only [4]

$$p-p_0=p_1+p_2,\quad \rho-\rho_0=\rho_1+\rho_2,\quad \mathbf{u}=\mathbf{u}_1+\mathbf{u}_2,\quad T-T_0=T_1+T_2. \tag{8}$$

On substituting (8) in (7), the equation of the first order can be obtained in the form

$$\begin{cases}\rho_0\dfrac{\partial\overline{u}_1}{\partial t}+\dfrac{\partial\overline{p}_1}{\partial x}=-\dfrac{1}{dy}\tau_{W1}\\[2mm] \dfrac{\partial\overline{p}_1}{\partial t}+\rho_0c_0^2\dfrac{\partial\overline{u}_1}{\partial x}=\dfrac{1}{dy}q_{W1}\end{cases} \tag{9}$$

where $\overline{u}_1=\dfrac{1}{dy}\displaystyle\int_{h-dy}^{h}u_1dy$, $\tau_{W1}=\mu\dfrac{\partial u_1}{\partial y}\Big|_W$, $q_{W1}=\lambda\dfrac{\partial T_1}{\partial y}\Big|_W$.

Where it has been taken into account that $\kappa p_0=\rho_0c_0^2$. These equations describe oscillation of the velocity and thermodynamic values in a row of cells situated near the wall of a long plane parallel channel. (9) contains unknown functions that can be approximated in the thin boundary layer approximation. Solution of (1) in the first (acoustic) approximation was given first by Tijdeman [5]. In assumption of high-frequency oscillation ($\delta << h$), one can write:

$$p_1=p(x)\,e^{i\omega t},u_1=u(x)\left[1-e^{-(1+i)\eta}\right]e^{i\omega t},\ T_1=\frac{p_1(x)}{\rho_0c_p}\left[1-(\gamma-1)e^{-(1+i)\eta\sqrt{\sigma}}\right]e^{i\omega t},$$
$$\rho_1=\frac{p_1(x)}{c_0^2}\left[1+(\gamma-1)e^{-(1+i)\eta\sqrt{\sigma}}\right]e^{i\omega t}\ \eta=\left(h-|y|\right)/\delta,\ \sigma=\mathrm{Pr}. \tag{10}$$

This approximation allows to calculate τ_{W1}, q_{W1}:

$$\tau_{w1}=\rho_0u_1(x)(1+i)\sqrt{\nu\omega/2}\,\exp i\omega t,$$
$$q_{w1}=-(1+i)p_1(x)\sqrt{a\omega/2}\,\exp i\omega t. \tag{11}$$

where $a=\nu/\mathrm{Pr}$ is thermo-diffusivity coefficient. Substituting (11) in (8) yields, assuming $dy=n\delta$:

$$\rho_0u_1(x)\left\{i\omega+(1+i)\frac{\sqrt{\nu\omega}}{\sqrt{2}n\delta}\right\}+\frac{dp_1(x)}{dx}=0,$$
$$p_1(x)\left\{i\omega+(1+i)(\gamma-1)\frac{\sqrt{a\omega}}{\sqrt{2}n\delta}\right\}+\rho_0c_0^2\frac{du_1(x)}{dx}=0. \tag{12}$$

To solve (12) by FDTD method one should write them in the discretized form:

$$(1+\frac{1}{2n})\rho_0\frac{u^{l+\frac{1}{2}}(i,j)-u^{l-\frac{1}{2}}(i,j)}{\delta t}=-\frac{p^l(i,j)-p^l(i-1,j)}{\delta x}-\frac{\rho_0\omega}{2n}u^l(i,j);$$
$$\left(1+\frac{\gamma-1}{2n\sqrt{\sigma}}\right)\frac{p^{l+1}(i,j)-p^l(i,j)}{\delta t}=-\omega\frac{\gamma-1}{2n\sqrt{\sigma}}p^{l+\frac{1}{2}}(i,j)-\rho_0c_0^2\frac{v^{l+\frac{1}{2}}(i,j)-v^{l+\frac{1}{2}}(i-1,j)}{\delta x}. \quad (13)$$

where the subscript l indicates discrete time, i and k – the spatial points, δx - spatial discretization step and δt - the time discretization step. After some simple transformations one finds:

$$C_1u^{l+\frac{1}{2}}(i,k)=C_2u^{l-\frac{1}{2}}(i,k)-\frac{p^l(i,k)-p^l(i,k-1)}{\delta x},$$
$$D_1p^{l+1}(i,k)=D_2p^l(i,k)-\rho_0c_0^2\frac{u^{l+\frac{1}{2}}(i,k)-u^{l+\frac{1}{2}}(i,k-1)}{\delta x}, \quad (14)$$

where $C_1=\left(\frac{\rho_0}{\delta t}\left(1+\frac{1}{2n}\right)+\frac{\rho_0\omega}{4n}\right)$, $C_2=\left(\frac{\rho_0}{\delta t}\left(1+\frac{1}{2n}\right)-\frac{\rho_0\omega}{4n}\right)$,

$$D_1=\left[\frac{1}{\delta t}+\frac{\gamma-1}{2n\sqrt{\sigma}}\left(\frac{1}{\delta t}+\omega\right)\right],\; D_2=\left[\frac{1}{\delta t}+\frac{\gamma-1}{2n\sqrt{\sigma}}\left(\frac{1}{\delta t}-\omega\right)\right].$$

The thermodynamic quantities ρ_1 and T_1 in acoustic approximation can be obtained from the third and the forth equations in (1) after substituting (8) in them

$$\rho_1=\frac{1}{R_gT_0}\{p_1-\rho_0R_gT_1\} \quad (15)$$

$$\rho_0c_p\frac{\partial T_1}{\partial t}-\frac{\partial p_1}{\partial t}=\lambda\frac{\partial^2T_1}{\partial y^2} \quad (16)$$

Averaging (16) over the cross-section of a cell yields:

$$\frac{\partial\overline{T_1}}{\partial t}=\frac{1}{\rho_0c_p}\frac{\partial p_1}{\partial t}+\frac{1}{\rho_0c_p}\frac{q_w}{n\delta} \quad (17)$$

where with the help of (10)

$$\frac{q_w}{\rho_0c_pn\delta}=-(1+i)\frac{(\gamma-1)}{2n\sqrt{\sigma}}\omega p_1(x)e^{i\omega t}.$$

Thus, (17) becomes

$$i\omega T_1(x)=i\omega p_1(x)\left(\frac{1}{\rho_0c_p}-\frac{\gamma-1}{2n\sqrt{\sigma}}\right)-\frac{\gamma-1}{2n\sqrt{\sigma}}\omega p_1(x) \quad (18)$$

Discretization of (18) yields

$$T_1^{l+1}(i,j)=T_1^l(i,j)+G_1p_1^{l+1}(i,j)-G_2p_1^l(i,j) \quad (19)$$

where

$$G_1=\left(\frac{1}{\rho_0c_p}-\frac{\gamma-1}{2n\sqrt{\sigma}}\left(1+\frac{\omega\delta t}{2}\right)\right),\; G_2=\left(\frac{1}{\rho_0c_p}-\frac{\gamma-1}{2n\sqrt{\sigma}}\left(1-\frac{\omega\delta t}{2}\right)\right)$$

Discretization of (15) results in

$$\rho_1^{l+1}(i,j) = \frac{1}{R_g T_0}\left\{p_1^{l+1}(i,j) - \rho_0 R_g T_1^{l+1}(i,j)\right\} \tag{20}$$

SECONDARY STREAMING SIMULATION

Secondary streaming is secondary steady motion and can be obtained by solving time-averaged second order Navier-Stokes equations. They are given here in the form they are used in simulations

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\rho_0\left\langle u_2^2\right\rangle\right)+\frac{\partial}{\partial y}\left(\rho_0\left\langle u_2 v_2\right\rangle\right)+\frac{\partial\left\langle p_2\right\rangle}{\partial x}-\frac{2}{3}\frac{\partial}{\partial x}\left(2\frac{\partial\left\langle\mu u_2\right\rangle}{\partial x}-\frac{\partial\left\langle\mu v_2\right\rangle}{\partial y}\right) \\
&\qquad -\frac{\partial}{\partial y}\left(\frac{\partial\left\langle\mu u_2\right\rangle}{\partial y}-\frac{\partial\left\langle\mu v_2\right\rangle}{\partial x}\right)=\left\langle F_x\right\rangle,
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial\left(\rho u_2 v_2\right)}{\partial x}+\frac{\partial\left(\rho v_2^2\right)}{\partial y}+\frac{\partial\left\langle p_2\right\rangle}{\partial y}-\frac{\partial}{\partial x}\left(\frac{\partial\left\langle\mu v_2\right\rangle}{\partial x}+\frac{\partial\left\langle\mu u_2\right\rangle}{\partial y}\right) \\
&\qquad -\frac{\partial}{\partial y}\left[\frac{2}{3}\left(2\frac{\partial\left\langle\mu v_2\right\rangle}{\partial y}-\frac{\partial\left\langle\mu u_2\right\rangle}{\partial x}\right)\right]=\left\langle F_y\right\rangle,
\end{aligned} \tag{21}$$

$$\frac{\partial}{\partial x}\left(\left\langle\rho_0 u_2\right\rangle\right)+\frac{\partial}{\partial y}\left(\left\langle\rho_0 v_2\right\rangle\right)=\left\langle S_m\right\rangle,$$

Here angular brackets mean the time averaging. The terms of the second order F_x, F_r and S_m entering the right side are the result of the interaction of the first order terms:

$$F_x = -\rho_0\left\langle u_1\frac{\partial u_1}{\partial x}\right\rangle-\rho_0\left\langle v_1\frac{\partial u_1}{\partial y}\right\rangle-\left\langle\rho_1\frac{\partial u_1}{\partial t}\right\rangle, \tag{22.a}$$

$$F_r = 0, \tag{22.b}$$

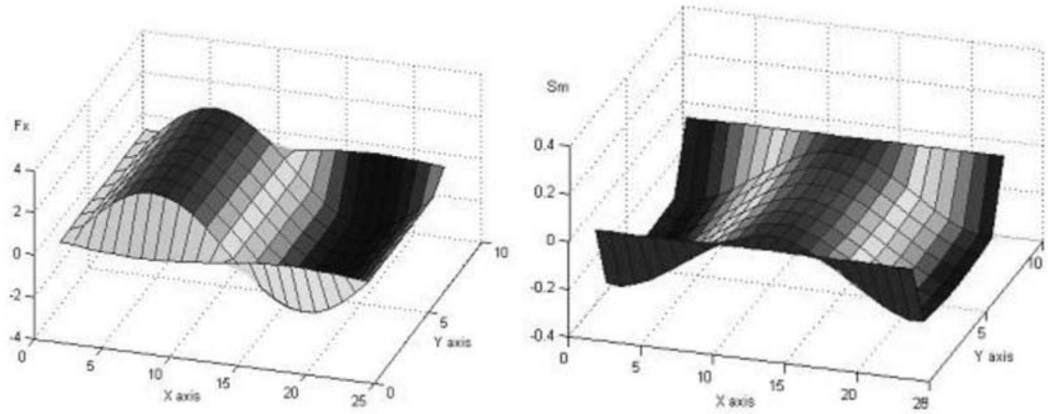
$$S_m = -\left\langle v_1\frac{\partial\rho_1}{\partial y}\right\rangle-\left\langle u_1\frac{\partial\rho_1}{\partial x}\right\rangle-\left\langle\frac{\rho_1}{\rho_0}\frac{\partial\rho_1}{\partial t}\right\rangle \tag{22.c}$$

RESULTS AND DISCUSSION

Parallel plate geometry

Let us consider a parallel plate channel of the length $L = 0.576\,m$ and the width $d = 0.026\,m$ closed with a sound source at the one end, the other one being closed by a rigid wall. A sound source is given in the form of pressure oscillation

$p(t) = A\cos\omega t$, where A is the amplitude. Simulation was carried out for the resonant frequency 295 Hz and $A = 1$.
For all cells nearest to the longitudinal walls, equations (14), (19), (20) were solved and, for the rest of cells, linearized Euler equations were solved [2]. Obtained matrices of values of u_1, v_1 and ρ_1 were used to calculate the nonlinear sources according to (22) within FDTD model.
Obtained values of the source terms approach acute ones overall except the cells with the boundary layer where they take mean values. To obtain the acute form of the sources, one should take dependence of transverse coordinate within the boundary layer into account, i.e. multiply all values by $(1-\exp(-\eta))$ (where $\eta = (0.5d - |0.5d - y|)/\delta$ is the dimensionless transverse coordinate). In fig.1 one can see the nonlinear sources in a parallel plate channel as a function of coordinates.



They can be approximated by the following functions
$F_x = -2.3 \cdot \sin(10.67x + 3.1)(1 - \exp(-\eta))$
$S_m = 0.32 \cdot (1 + \cos(10.67x + 3.1))(1 - \exp(-\eta))$.
Equations (21) were solved with the help of CFD technique realized in Fluent software. In fig.2, the streamlines are given in a parallel plate channel

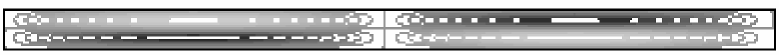


Figure 2 - Streamlines in a parallel plate channel

Parallel plate geometry with a stack

Analogous procedure was carried out for the same geometry with addition of ten plates in the middle of a channel. The assumption was made that the plates have a negligible thickness. Equations (14), (15), (19) had to be solved in the cells from both sides of plates, i.e. in two neighbour rows of cells, one row of cells being left for nonviscous fluid between two viscous ones. In fig. 3, the nonlinear source terms are given as functions of coordinates.

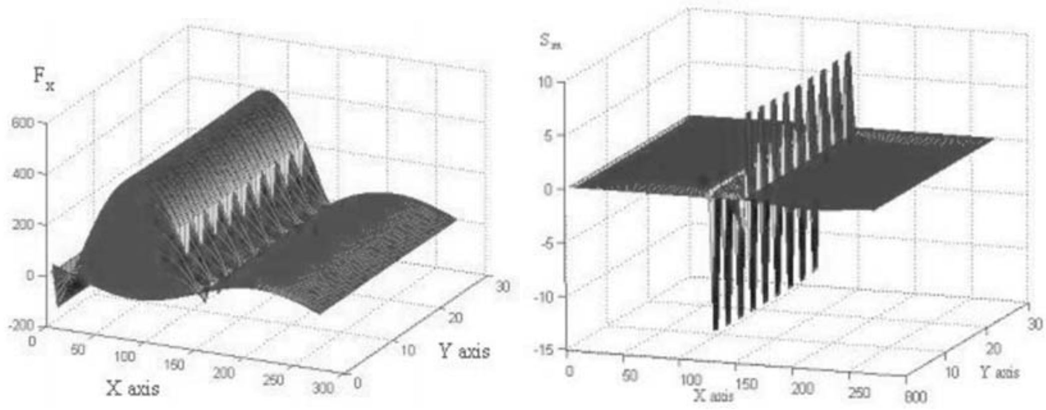


Figure 3 – The nonlinear sources F_x and S_m as functions of coordinates.

CONCLUSIONS

A numerical model of acoustic oscillation in a parallel plate channel filled by a viscous and heat conducting gas is suggested that includes a subgrid-scale analytical approximation for the boundary layers within the framework of the FDTD discretization. The obtained solution was used to calculate nonlinear source terms entering the stationary second order equations. The calculated nonlinear sources were used in simulation of acoustic streaming carried out numerically by virtue of standard CFD technique. For a parallel plate channel with a stack consisted of ten plates, the nonlinear sources were obtained that are to be used to calculate acoustic streaming. Streamlines were obtained for parallel plate geometry. Results of the simulation agree with known experimental and analytical data.

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